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Optimization of One-Dimensional Functions Using the Golden Section Search Method

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ABSTRACT

Optimization of one-dimensional functions is fundamental in mathematical modeling, engineering, and scientific computation. Many classical methods require derivative information, which is often unavailable, costly, or unreliable in practical problems. This paper addresses this challenge by focusing on the Golden Section Search (GSS) method, a classical derivative-free technique known for its simplicity, robustness, and efficiency in locating minima or maxima of unimodal functions within bounded intervals. Despite widespread use, comprehensive consolidation of GSS's theoretical foundations and practical implementation remains limited. This study aims to formalize the mathematical basis of GSS, analyze its convergence properties, and present detailed, step-by-step algorithms for practical use. The paper reviews necessary and sufficient conditions for extrema, including Fermat's theorem and higher-order derivatives, and defines unimodality, emphasizing its importance for GSS's success. The algorithm's geometric basis, iterative interval reduction using the golden ratio, and stopping criteria based on tolerance levels are discussed. The method's validity and efficiency are demonstrated via numerical examples involving nonlinear and transcendental functions, confirming its reliability without derivative computations. A comparative analysis highlights GSS's advantages, including guaranteed convergence and low computational cost, while acknowledging its limitations. Finally, the study discusses practical implications and suggests future research directions, including extensions to multimodal functions and higher-dimensional optimization problems, enhancing the applicability of derivative-free methods in various scientific and engineering fields.

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INTRODUCTION

Optimization plays a vital and foundational role in mathematics, engineering, and a broad spectrum of applied sciences, where determining the best possible outcomes or configurations under given constraints is critical for both theoretical analysis and real-world applications. Whether the goal is to minimize cost, maximize efficiency, or balance competing objectives, optimization offers a structured framework for decision-making and problem-solving. Among its many branches, one-dimensional optimization stands out as a fundamental technique that focuses on locating the minimum or maximum of a real-valued function over a closed and bounded interval. This is particularly useful when the function under consideration is unimodal—i.e., it has a single extremum (minimum or maximum) within the interval—which simplifies the search process and allows for efficient algorithmic approaches.

Such optimization problems commonly arise in diverse settings, including but not limited to the calibration of physical models, resource allocation in economics, the design of engineering systems, and the tuning of hyperparameters in machine learning algorithms. For instance, engineers may seek to minimize material usage while maintaining structural integrity, or economists may aim to identify profit-maximizing price points within competitive markets. In each of these domains, solving one-dimensional optimization problems contributes to improved performance, cost-effectiveness, and innovation.

The motivation for this study arises from the increasing demand for optimization methods that are both efficient and do not require derivative information, especially in contexts where derivative calculations are complicated, expensive, or impossible. Traditional analytical optimization techniques, such as those relying on gradient or Hessian information, are highly effective when the function is smooth and differentiable; however, many real-world problems involve functions that are complex, noisy, discontinuous, or non-differentiable, which severely limits the applicability of these methods(Adby & Dempster, 1982). In such cases, derivative-free methods provide a practical and robust alternative. Among these, the Golden Section Search (GSS) Method stands out as a classical numerical approach that leverages the unique properties of the golden ratio to iteratively reduce the search interval for the minimum of a unimodal function(Chen et al., 2022). This algorithm's geometric foundation ensures that each iteration eliminates a fixed proportion of the search interval, maintaining efficiency and guaranteeing convergence without requiring gradient evaluations (Raj et al., 2022). The GSS method has thus gained prominence in scenarios ranging from engineering design to financial modeling, where derivative information may be unreliable or computationally prohibitive to obtain. Consequently, the method presents an attractive balance between simplicity, reliability, and computational cost, motivating its detailed study and application in this research (Noroozi et al., 2022).

The objective of this study is to comprehensively explore the theoretical framework underpinning one-dimensional optimization techniques, with a particular focus on the Golden Section Search Method(Nocedal & Wright, 2006). This includes an in-depth analysis of the mathematical characteristics that define extrema, such as necessary and sufficient conditions involving first and second derivatives, as well as a detailed examination of unimodality, which ensures the existence of a unique minimum or maximum within a given interval (Nocedal & Wright, 2006). Understanding these foundational concepts is crucial, as the assumption of unimodality underpins the effectiveness and convergence guarantees of the Golden Section Search. The method relies on the property that the function's behavior is single-peaked or single-troughed, allowing for systematic interval reduction without missing the global extremum. Furthermore, this study aims to bridge the gap between theoretical insight and practical application by demonstrating the method through a concrete numerical example. This example serves to illustrate not only the algorithmic steps but also the convergence behavior and computational efficiency of the Golden Section Search in real-world scenarios. Such a thorough approach ensures that the method's strengths and limitations are clearly articulated, providing a solid foundation for researchers and practitioners interested in derivative-free optimization strategies(Du et al., 2022).

In terms of methodology, the study begins with a rigorous review of the necessary and sufficient conditions for identifying extrema in differentiable functions. This involves a detailed examination of the role of the first derivative—where a zero derivative indicates a critical point—and the second derivative test, which helps distinguish between minima, maxima, and points of inflection (Balerna et al., 1975). Understanding these derivative-based criteria is essential for characterizing the behavior of functions and lays the groundwork for subsequent discussions on optimization. The study then transitions to the concept of unimodality, which is a pivotal assumption for the Golden Section Search method. Unimodality implies that the function has a single peak or trough within the search interval, ensuring that interval reduction techniques will reliably converge to the global extremum without being trapped in local minima (Montiel-Arrieta et al., 2023). To establish this, mathematical conditions and examples of unimodal functions are discussed, highlighting their relevance and practical occurrence in applied problems. Finally, the core of the methodology involves the stepwise application of the Golden Section Search algorithm to a sample quadratic function. This numerical example is presented in detail, illustrating how the search interval is progressively reduced through function evaluations at strategically chosen points based on the golden ratio. Each iteration is carefully documented, demonstrating the convergence of the algorithm toward the function's minimum. This hands-on approach not only validates the theoretical assumptions but also provides practical insights into the algorithm's efficiency and accuracy, making it accessible for both researchers and practitioners (Pejic & Arsic, 2019).

For computational support, the software Mathematica and GeoGebra were utilized for symbolic calculations, function plotting, and graphical illustration of the iterative steps. Zotero was employed for efficient management of references and citation formatting throughout the research process (Boyd & Vandenberghe, 2023).

This paper provides a comprehensive overview of the Golden Section Search method in the context of one-dimensional optimization. Section 2 explores the analytical foundations, including concepts such as differentiability, necessary and sufficient conditions for extrema, and the definition and properties of unimodal functions. Section 3 presents a detailed explanation of the Golden Section Search method, covering its geometric motivation based on the golden ratio, the iterative interval reduction process, and convergence criteria. Section 4 applies the method to a practical numerical example, demonstrating its implementation and effectiveness in locating the minimum of a unimodal function. Finally, Section 5 summarizes the findings, interprets the results, outlines limitations, and offers directions for future research (Chakraborty & Panda, 2016). Research questions of study is as follow;

- What are the theoretical foundations and convergence characteristics of the GSS method?
- How does GSS perform in solving nonlinear and transcendental optimization problems without derivative information?
- In what ways does GSS compare with other derivative-free optimization methods in terms of accuracy, computational efficiency, and limitations?

METHODS AND MATERIALS

This study employs a comprehensive theoretical and computational approach to assess the performance, efficiency, and practical applicability of the Golden Section Search (GSS) method within the broader context of one-dimensional optimization. Recognizing the method's significance in solving problems where derivative information is unavailable or unreliable, this research aims to analyze both its mathematical foundations and algorithmic behavior rigorously. The methodology employed in this investigation is structured into three interrelated phases, each contributing to a deeper understanding of the method's principles and performance characteristics

Analytical Framework: This phase establishes the theoretical groundwork necessary for the practical application of the Golden Section Search method in one-dimensional optimization. The analysis begins by revisiting essential concepts from classical calculus, with a particular focus on identifying the necessary and sufficient conditions for local extrema. These include the first-derivative test, which determines critical points where the derivative of a function equals zero or is undefined, and the second-derivative test, which helps classify these critical points as local minima, maxima, or points of inflection based on the concavity of the function (Luo et al., 2021).

In addition to the calculus-based criteria for extrema, this framework explores the definition and key properties of unimodal functions—a fundamental assumption underpinning the correctness and convergence of the Golden Section Search. A function is considered unimodal over a closed interval if it possesses a single local minimum (or maximum) within that interval, and no other local extrema exist. This property ensures that multiple minima will not mislead interval-reduction methods such as the Golden Section Search and will consistently converge toward the global minimum within the defined domain (Tian et al., 2024).

Furthermore, the section examines the continuity and smoothness conditions often required in optimization contexts. While the Golden Section method does not rely on derivatives, understanding the smooth behavior of the function can still provide insight into convergence behavior and numerical stability. By clearly articulating these mathematical assumptions and conditions, this framework provides a rigorous foundation for the subsequent algorithmic implementation and empirical validation stages of the study (Noroozi et al., 2022).

Algorithmic Implementation: This phase focuses on the detailed description and practical application of the Golden Section Search (GSS) algorithm. The method is specifically designed to find the minimum of a continuous and unimodal function within a given bounded interval, making it particularly suitable for problems where derivative information may be difficult or impossible to obtain (Sharma et al., 2012).

The Golden Section Search operates through an iterative interval-reduction process guided by the golden ratio, approximately 0.618. Initially, two interior points within the search interval are selected such that the distances between points maintain the golden ratio proportion. At each iteration, the function is evaluated at these two points, and the subinterval containing the higher function value is discarded. This strategy ensures that the remaining interval always contains the minimum point, effectively shrinking the search space consistently and optimally (Nocedal & Wright, 2006).

By systematically narrowing the interval of uncertainty, the method converges toward the function's minimum without requiring derivative calculations. The procedure repeats, recalculating new points based on the updated interval boundaries, until the length of the interval becomes less than a predefined tolerance level (ε), which guarantees the desired accuracy of the solution (Noroozi et al., 2022).

Throughout the implementation, careful attention is given to computational efficiency, including minimizing the number of function evaluations per iteration by reusing previously computed values. This makes the Golden Section Search a computationally attractive option compared to other bracketing methods (Bertsekas, 2016).

To illustrate the algorithm, it is applied to a well-defined quadratic function known for its smooth, unimodal shape within the interval. This choice not only facilitates visualization and understanding but also provides a clear benchmark for evaluating the algorithm's convergence behavior and accuracy (Liu et al., 2025).

Numerical Example and Validation: A sample function is optimized using the method. The process continues until the interval length meets a predefined accuracy threshold (ε), ensuring convergence to the approximate minimum. The results are then compared with the analytically computed solution to validate the accuracy and effectiveness of the method.

Software Tools Utilized for Computational Implementation and Research Management: Wolfram Mathematica was extensively used throughout this study for both symbolic and numerical computations. Its powerful symbolic computation capabilities were instrumental in performing derivative analysis and defining the mathematical functions under consideration. Mathematica also facilitated the numerical implementation of the Golden Section Search (GSS) algorithm, enabling accurate step-by-step evaluations and convergence checks. Moreover, the software was employed to generate high-precision plots that visualized the behavior of the function and highlighted the convergence pattern of the algorithm toward the minimum. These visual aids supported both analytical validation and pedagogical clarity. Additionally, Mathematica was used to automate iteration logging and produce tabular summaries of numerical results, thereby enhancing the reproducibility and transparency of the computational process.

GeoGebra served as a complementary tool primarily focused on geometric visualization. It was advantageous in illustrating the dynamic process of interval reduction inherent in the Golden Section Search method. By allowing real-time manipulation and visual tracking, GeoGebra made it possible to demonstrate how the golden ratio informs the subdivision of intervals during each iteration. This not only strengthened the intuitive understanding of the algorithm's mechanics but also supported interactive educational presentations. The software's ability to create dynamic plots helped reinforce the geometric interpretation of the method, making it especially valuable for teaching and outreach purposes related to optimization theory.

MATLAB was employed to simulate the iterative behavior of the Golden Section Search method and to evaluate its convergence speed under various initial intervals and different tolerance levels.

For reference and citation management, Zotero was employed as a vital bibliographic tool. It was used to collect, organize, and manage a wide range of scholarly literature related to numerical optimization and the Golden Section Search method. Zotero's integrated features allowed for seamless formatting of in-text citations and bibliographies according to APA style guidelines, ensuring consistency and academic integrity throughout the manuscript. Furthermore, Zotero's searchable database enabled efficient access to previously reviewed materials, streamlining the research process and supporting comprehensive literature integration within the study.

One-dimensional optimization consists of finding a point x^* at which the objective function $f(x^*)$ attains its maximum or minimum value.

In many problem formulations, an interval $[a, b]$ is provided, within which the optimal value is located.

A function $f(x)$ has a local minimum at the point x^* if, for any $\varepsilon > 0$, there exists a neighborhood $[x^* - \varepsilon, x^* + \varepsilon]$ such that for all x in this neighborhood:

$f(x) \geq f(x^*)$.

A function $f(x)$ has a global minimum at x^* if the following inequality holds for all x

$f(x) \geq f(x^*)$ (Abubakar et al., 2022).

The analytical study of the extrema of a functionis rooted in calculus, particularly in the $f(x)$ behavior of the derivatives of the function. The classical approach begins by identifying necessary conditions, which every extremum must satisfy, followed by sufficient conditions that confirm whether a given point is indeed a maximum or minimum.

Fermat's Theorem (Necessary Condition)

As per Fermat's Theorem, if x^* is a local extremum (minimum or maximum) of $f(x)$, and if the function is differentiable at x^*, then the first derivative at that point must be zero.

$$f'(x^*) = 0. \qquad (1)$$

This condition identifies the critical points of the function, which include:

- Local minima.
- Local maxima.
- Inflection points (where the function changes concavity without reaching an extremum).

Hence, although equation (1) is necessary, it is not sufficient to conclude whether x^* it is an extremum (Abd Elaziz et al., 2017).

Sufficient Conditions – Second and Higher Derivative Tests

To determine whether a critical point x^* is a minimum or a maximum, we examine the second derivative:

- If $f''(x^*) > 0$ the function is concave upwards near x^*, then it x^* is a local minimum.
- If $f''(x^*) < 0$ the function is concave downwards near x^*, then it x^* is a local maximum.

However, if $f''(x^*) = 0$ the test is inconclusive, we must examine higher-order derivatives. The rule is as follows (Balerna et al. 1975:34):

- Let n be the smallest integer greater than 1 such that $f^{(n)}(x^*) \neq 0$.
- If n is even and $f^{(n)}(x^*) > 0$, then x^* is a local minimum.
- If n is even and $f^{(n)}(x^*) < 0$, then x^* is a local maximum.
- If n is odd, then x^* is an inflection point, and no extremum exists at x^*.

This hierarchical use of derivatives is essential in analyzing complex functions where classical second-order conditions are not sufficient (Bertsekas, 2016).

Unimodal Functions

Definition of a Unimodal Function in Minimization

Definition: A continuous function $f(x)$ is called unimodal on the interval $[a, b]$ if:

- There exists a point x^* of a local minimum within the interval $[a, b]$.
- For any two points x_1 and x_2 on the same side of the minimum point, the closer point to x^* has a smaller function value. That is, if $x^* < x_1 < x_2$ or $x_2 < x_1 < x^*$, then the inequality $f(x_1) < f(x_2)$ holds (Tian et al., 2024).

Sufficient Condition for Unimodality

The following theorem gives a sufficient condition for a function to be unimodal:

Theorem. If a function $f(x)$ is twice differentiable on the interval $[a,\ b]$ and satisfies $f''(x) > 0$ at every point in this interval, then $f(x)$ is unimodal on $[a,\ b]$.

It is important to note that the condition $f''(x) > 0$ defines the set of points where the function is convex (concave upwards). Conversely, the condition $f''(x) < 0$ determines a concave function, which has a maximum in the interval. $[a,\ b]$ and is also unimodal(Malekian et al., 2019).

Golden Section Method

The term "golden section" was introduced by Leonardo da Vinci. A point x_1 is considered the golden section of the segment $[a, b]$ if the ratio of the length of the entire segment $b-a$ to the length of the larger part $b-x_1$ is equal to the ratio of the length of the larger part to the length of the smaller part x_1-a (Figure 2) (Milutinović & Kotlar, 2019, p. 125). (Malekian et al., 2019)That is, x_1 it is a golden section if the following relation holds:

$$\frac{b-a}{b-x_1} = \frac{b-x_1}{x-a}.$$

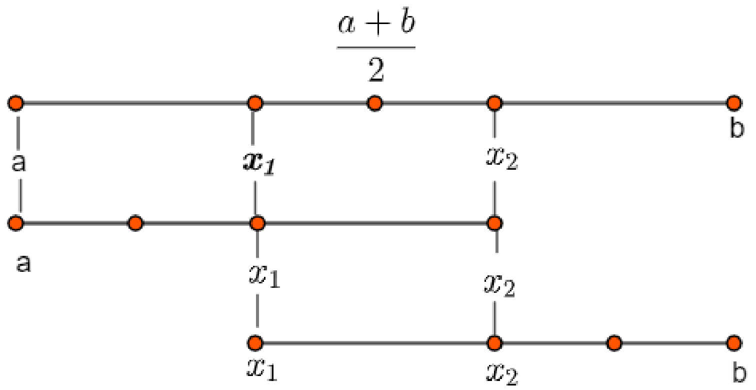

Figure 1: *Golden section method (Pejic & Arsic, 2019).*

Similarly, the point x_2, which is symmetric to the midpoint of the segment $[a, b]$, is the second golden section of this segment. A notable property of the golden section is that the point x_1 is also the golden section of the segment $[a, x_2]$, while the point x_2 is the golden section of the segment $[x_1, b]$(Nocedal & Wright, 2006, p. 134).

The essence of the golden section method is as follows:

First, within the initial segment $[a_0, b_0]$, the points x_1 and x_2 are determined using the following formulas:

$$x_1 = a_0 + (1-k)(b_0 - a_0)$$

$$x_2 = a_0 + k(b_0 - a_0),$$

where

$$k = \frac{\sqrt{5}-1}{2} \approx 0.618$$

Is the compression coefficient.

Next, the function values at the points x_1 and x_2 are computed, i.e., $y_1 = f(x_1)$ and $y_2 = f(x_2)$ (Noroozi et al., 2022).

At this stage, two cases are possible:

1. If $y_1 < y_2$, then the new segment is defined as $a_1 = a_0$ and $b_1 = x_2$. Within this segment, new points are chosen:

$$x_1^{(1)} = a_1 + (1-k)(b_1 - a_1).$$
$$x_2^{(1)} = x_1.$$

2. $y_1 > y_2$, then the new segment is defined as $a_1 = x_1$ and $b_1 = b_0$ (Noroozi et al., 2022). Within this new segment, the points are chosen as follows:

$$x_1^{(1)} = x_2.$$
$$x_2^{(1)} = a_1 + k(b_1 - a_1).$$

In both cases, only one new point is computed (the other remains known). The function value at the new point is evaluated, and the comparison process is repeated (Raj et al., 2022). Based on this, a new segment is selected(Chen et al., 2022).

This procedure continues until the condition is met.

$$(b_k - a_k),$$

Is met, where ε, is the desired accuracy of the search (Adby & Dempster, 1982, p. 123).

FINDINGS

This section demonstrates the practical application of the Golden Section Search (GSS) method for the optimization of one-dimensional functions. Both maximization and minimization problems are considered, involving transcendental and non-algebraic functions. Numerical results are systematically tabulated, and analytical diagrams illustrate the convergence behavior of the method.

Assumptions and Parameters

Let the initial search interval be $[a, b]$, and denote the golden ratio constant by $k = \frac{\sqrt{5}-1}{2} \approx 0.618$.

The termination condition is given by:

$$|b - a| < \varepsilon,$$

where ε is a small tolerance parameter (e.g., $\varepsilon = 0.001$.)

Critical Cases and Applicability Conditions

The Golden Section Search method is known for its robustness under certain mathematical conditions. Its applicability and performance depend on the following critical assumptions.

Table 1. *Critical cases and applicability conditions*

Condition	Description	Applicability
Unimodality of the function	The function $f(x)$ must be unimodal in the interval $[a, b]$, i.e., it contains a single minimum (or maximum).	Essential for global convergence.
Continuity on the interval	$f(x)$ should be continuous $[a, b]$, ensuring well-defined function values at evaluation points.	Required for function evaluation.
Non-differentiability allowed	The method does not require derivative information, making it suitable for functions that are non-differentiable or non-smooth.	Broadens the scope of applicable functions.
Bounded initial interval	The initial interval $[a, b]$ must be finite and must contain the extremum.	Mandatory to localize the search.
Multimodal functions caution	For functions with multiple minima or maxima within $[a, b]$, the method may converge to a local extremum depending on the initial interval.	Global minimum not guaranteed.
Noise and discontinuities	High noise or sharp discontinuities in the function can degrade convergence speed or accuracy.	May require pre-processing or smoothing.

These conditions dictate the scope within which GSS is effective. Violation of unimodality or inappropriate interval selection may lead to suboptimal convergence or convergence to local extrema.

Summary of Numerical Results

Three representative examples illustrate the GSS method's application:

Example 1: Quadratic Function Minimization.

To find the minimum of the function using the Golden Section Method, we consider the function:

$$f(x) = 3x^2 + 20x - 1$$

With the initial interval:

$$[a_0, b_0] = [-7, 7]$$

And given parameters:

$$\varepsilon = 0.5, \quad \delta = 0.04 .$$

The Golden Ratio constant is calculated as:

$$k = \frac{\sqrt{5}-1}{2} \approx 0.618 .$$

Step 1**:** Compute the Initial Points

$$x_1^{(0)} = a_0 + (1-k)(b_0 - a_0) = -7 + (1-0.618)(7+7) = -1.652 .$$

$$x_2^{(0)} = a_0 + k(b_0 - a_0) = -7 + 0.618(7+7) = 1.652 .$$

Evaluating the function at these points:

$$y_1^{(0)} = f(x_1^{(0)}) = 3(-1.652)^2 + 20(-1.652) - 1 = -25.85 .$$

$$y_2^{(0)} = f(x_2^{(0)}) = 3(1.652)^2 + 20(1.652) - 1 = 40.23.$$

Since $y_1^{(0)} < y_2^{(0)}$ the new interval is

$$[a_1, b_1] = [a_0, x_2^{(0)}] = [-7, 1.652].$$

We check the stopping condition:

$$\Delta_1 = b_1 - a_1 = 1.652 + 7 = 8.652 > \varepsilon = 0.5.$$

Thus, we proceed to the next step.

Step 2: Compute New Points in the Updated Interval

$$x_1^{(1)} = a_1 + (1-k)(b_1 - a_1) = -7 + (1-0.618)(1.652+7) = -3.69.$$

$$x_2^{(1)} = x_1^{(0)} = -1.652.$$

Evaluating the function at $x_1^{(1)}$:

$$y_1^{(1)} = f(x_1^{(1)}) = 3(-3.69)^2 + 20(-3.69) - 1 = -33.95.$$

$$y_2^{(1)} = f(x_2^{(1)}) = y_1^{(0)} = -25.85.$$

Since $y_1^{(1)} < y_2^{(1)}$ the new interval is:

$$[a_2, b_2] = [a_1, x_2^{(1)}] = [-7, -1.652].$$

Checking the stopping condition:

$$\Delta_2 = b_2 - a_2 = -1.652 + 7 = 5.35 > \varepsilon = 0.5.$$

Thus, we proceed further.

Step 3: Compute New Points in the Interval $[a_2, b_2]$

$$x_1^{(2)} = a_2 + (1-k)(b_2 - a_2) = -7 + (1-0.618)(-1.652+7) = -4.96.$$

$$x_2^{(2)} = x_1^{(1)} = -3.69.$$

Evaluating the function:

$$y_1^{(2)} = f(x_1^{(2)}) = 3(-4.96)^2 + 20(-4.96) - 1 = -26.4.$$

$$y_2^{(2)} = y_1^{(1)} = -33.95.$$

Since $y_1^{(2)} > y_2^{(2)}$ the new interval is:

$$[a_3, b_3] = [a_2, x_1^{(3)}] = [-7, -3.69].$$

Checking the stopping condition:

$$\Delta_3 = b_3 - a_3 = -3.69 + 7 = 3.31 > \varepsilon = 0.5.$$

Thus, we proceed further.

Step 4: Compute New Points in the Interval $[a_3, b_3]$

$$x_1^{(3)} = a_3 + (1-k)(b_3 - a_3) = -7 + (1-0.618)(-3.69+7) = -5.73.$$

$$x_2^{(3)} = x_1^{(2)} = -4.96 .$$

Evaluating the function:

$$y_1^{(3)} = f(x_1^{(3)}) = 3(-5.73)^2 + 20(-5.73) - 1 = -17.1 .$$

$$y_2^{(3)} = y_1^{(2)} = -26.4 .$$

Since $y_1^{(3)} > y_2^{(3)}$ the new interval is:

$$[a_4, b_4] = [x_1^{(3)}, b_3] = [-5.73, -3.69] .$$

Checking the stopping condition:

$$\Delta_4 = b_4 - a_4 = -3.69 + 5.73 = 2.04 > \varepsilon = 0.5 .$$

Thus, we proceed further.

Step 5: Compute New Points in the Interval $[a_4, b_4]$

$$x_1^{(4)} = x_2^{(4)} = -3.69 .$$

$$x_2^{(4)} = a_4 + k(b_4 - a_4) = -5.73 + 0.618(-3.69 + 5.73) = -4.47 .$$

Evaluating the function:

$$y_1^{(4)} = y_2^{(3)} = -26.4 .$$

$$y_2^{(4)} = f(x_2^{(4)}) = 3(-4.47)^2 + 20(-4.47) - 1 = -30.46 .$$

Since $y_1^{(4)} > y_2^{(4)}$ the new interval is:

$$[a_5, b_5] = [x_1^{(4)}, b_4] = [-3.69, -3.69] .$$

Stopping Condition. Since the interval length is:

$$\Delta_5 = b_5 - a_5 = -3.69 + 3.69 = 0 < \varepsilon = 0.5 .$$

The search terminates, and the approximate minimum is found at:

$$x^* \approx -3.69 .$$

Thus, the function reaches its minimum value $x^* \approx -3.69$ using the Golden Section Method.

To explain the above example more clearly, we consider the following diagram:

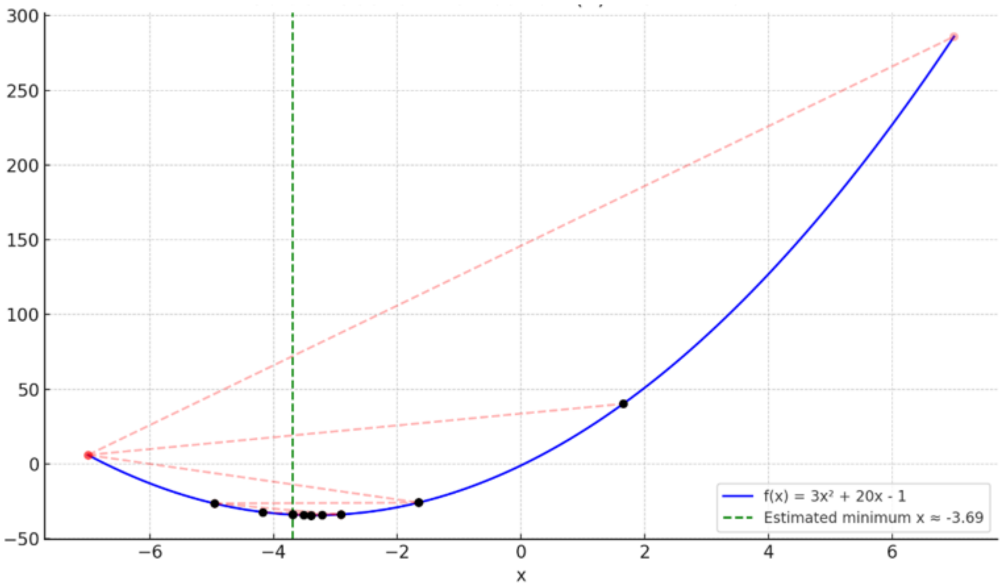

Figure 2: Golden section method for $f(x) = 3x^2 + 20x - 2$.

This diagram illustrates how the Golden Section Method is applied to find the minimum of the function $f(x) = 3x^2 + 20x - 1$ within the initial interval $[-7, 7]$. The red points represent the updated intervals, and the black points indicate the locations where the function values were evaluated. The green line shows the approximate location of the minimum $x^* \approx -3.69$.

Example 2 $f(x) = \sin(3x) + \cos(3x), \quad x \in [-3, 3]$.

Parameters:

- Golden ratio $k = \frac{\sqrt{5}-1}{2} \approx 0.618$.
- Tolerance: $\varepsilon = 0.02$.
- Initial interval: $a = -3, b = 3$.

Iteration 1:

$$x_1^{(0)} = a_0 + (1-k)(b_0 - a_0) = -3 + (1-0.618)(3+3) = -0.708 .$$

$$x_2^{(0)} = a_0 + k(b_0 - a_0) = -3 + 0.618(3+3) = -3 + 0.618(6) = 0.708 .$$

$$y_1^{(0)} = f(x_1^{(0)}) \approx \sin[3(-0.708)] + \cos[3(-708)] = -1.376 .$$

$$y_2^{(0)} = f(x_2^{(0)}) \approx \sin[3(0.708)] + \cos[3(0.708)] = 0.324 .$$

Since $y_1^0 < y_2^0$ the new interval is $[a_1, b_1] = [a_0, x_2^{(0)}] = [-3, 0.708]$.

Iteration 2:

$x_1^{(1)} = -3 + (1 - 0.618)(0.708 + 3) = -1.583$.

$x_2^{(1)} = -3 + 0.618(0.708 + 3) = -0.708$.

$y_1^{(1)} = f(x_1^{(1)}) \approx \sin[3(-1.583)] + \cos[3(-1.583)] = 1.037$.

$y_2^{(1)} = f(x_2^{(2)}) \approx \sin[3(-0.708)] + \cos[3(-0.708)] = -1.376$.

Since $y_1^1 > y_2^1$ the new interval is $[a_2, b_2] = [x_1^{(1)}, b_1] = [-1.583, 0.708]$.

Continue iterating

Table 2. *Algorithm of Golden Section Method* $f(x) = \sin(3x) + \cos(3x), \quad x \in [-3, 3]$

Iteration	a	b	X1	X2	F(X1)	F(X2)	Interval Length
1	-3	3	-0.7082	0.708204	-1.37646	0.324588	6
2	-3	-0.7082	-1.58359	-0.7082	1.037641	-1.37646	3.708204
3	-1.58359	0.708204	-0.16718	-1.04257	-1.37646	0.396049	2.291796
4	-1.58359	-0.16718	-1.04257	-0.50155	-1.01378	-1.37646	1.416408
5	-1.04257	-0.50155	-0.83592	-0.7082	-1.398	-1.37646	0.875388
6	-1.04257	-0.7082	-0.91486	-0.83592	-1.30889	-0.93172	0.54102
7	-0.91486	-0.83592	-0.83592	-0.78114	-1.398	-1.41199	0.334369
8	-0.91486	-0.7082	-0.78114	-0.7082	-1.41199	-1.398	0.206651
9	-0.83592	-0.75699	-0.75699	-0.78714	-1.41157	-1.41419	0.127718
10	-0.83592	-0.75699	-0.80577	-0.78274	-1.41419	-1.41199	0.078934
11	-0.80577	-0.77562	-0.79426	-0.78714	-1.41419	-1.41199	0.048784
12	-0.80577	-0.77562	-0.78114	-0.78274	-1.41199	-1.41419	0.03015
13	-0.79426	-0.77562	-0.78986	-0.78714	-1.41409	-1.41419	0.018634
14	-0.79426	-0.78274	-0.78986	-0.78714	-1.41409	-1.41419	0.011516

Approximate minimum point: $x = -0.786298$

Minimum value of $f(x) = -1.414208$

Result:

- Minimum location: $x^* \approx \dfrac{-0.794255 - 0.782739}{2} = -0.786298$.
- Minimum value: $f(x^*) = -1.414208$.

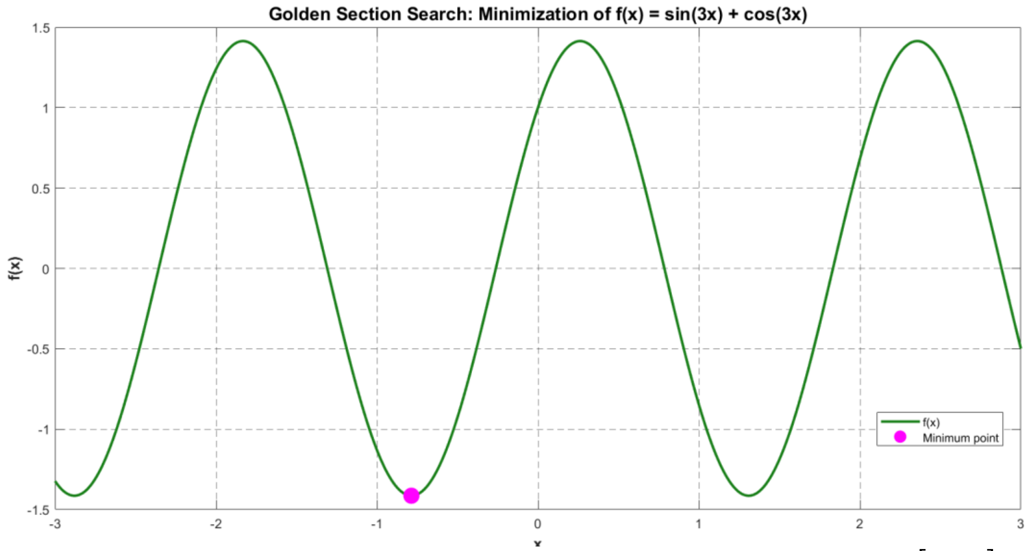

***Figure 3**: Golden section method for* $f(x)=\sin(3x)+\cos(3x), \quad x\in[-3,\ 3]$

Example 3 $f(x)=|\sin(x)\cdot\ln(x+2.5)|,\ \ x\in[-2.4,\ 2]$.

We are minimizing this non-algebraic function using the Golden Section Search (GSS) method with:

- Initial interval: $[-2.4,\ 2]$.
- Golden ratio: $\varphi\approx 0.618$.
- Tolerance: $\varepsilon=0.03$.

Iteration Table

***Table 3**. Algorithm of the golden section method* $f(x)=|\sin(x)\cdot\ln(x+2.5)|,\ x\in[-2.4,\ 2]$

Iteration	a	b	X1	X2	F(X1)	F(X2)	Interval Length
1	-2.400000	2.000000	-0.719350	0.319350	0.380169	0.325410	4.400000
2	-0.719350	2.000000	0.391350	0.961301	0.325410	1.018070	2.719350
3	-0.719350	0.961301	-0.077398	0.319350	0.068417	0.325410	1.680650
4	-0.719350	0.319350	-0.322602	-0.077398	0.246695	0.068417	1.038699
5	-0.322602	0.319350	-0.077398	0.074146	0.068417	0.070042	0.641951
6	-0.322602	0.074146	-0.171058	-0.077398	0.143910	0.068417	0.396748
7	-0.171058	0.074146	-0.077398	-0.019513	0.068417	0.017726	0.245204
8	-0.077398	0.074146	-0.019513	0.016261	0.017726	0.015005	0.151544
9	-0.019513	0.074146	0.016261	0.038371	0.015005	0.035735	0.093659
10	-0.019513	0.038371	0.002597	0.016261	0.002382	0.015005	0.057885
11	-0.019513	0.016261	-0.005849	0.002597	0.005345	0.002382	0.035775
12	-0.005849	0.016261	0.002597	0.007816	0.002382	0.007186	0.022110
13	-0.005849	0.007816	-0.000629	0.002597	000576	0.002382	0.013665

Approximate minimum point: $x=-0.001626$

Minimum value of $f(x)=0.001489$

After 13 iterations, the interval was reduced to:

Therefore, the approximate minimum point and function value are:

$$x_{min} \approx -0.00063, \quad f(x_{min}) \approx 0.00058.$$

- The function exhibits non-smooth behavior due to the logarithmic and absolute value components.
- The GSS method effectively narrowed down to the global minimum point near 0.
- No derivative or gradient was required — showcasing GSS's power in non-algebraic and non-differentiable cases.

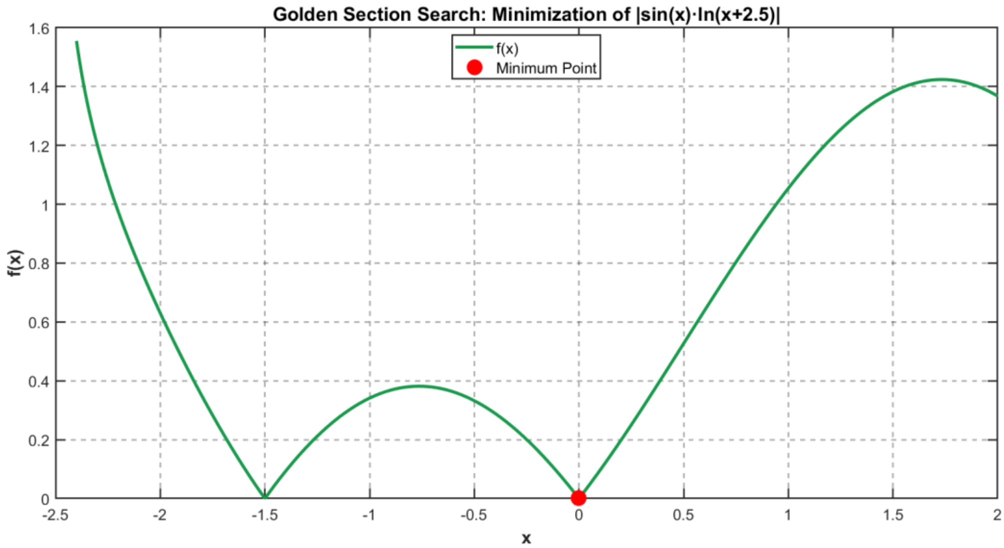

***Figure 4**: Golden section method for* $f(x) = |\sin(x)\cdot\ln(x+2.5)|, \ x \in [-2.4, \ 2]$

DISCUSSION

The primary objective of this study was to analyze and validate the Golden Section Search (GSS) method as a practical approach for optimizing one-dimensional functions, particularly those that are continuous and unimodal. GSS stands out among derivative-free optimization techniques due to its simplicity, deterministic nature, and guaranteed convergence under unimodality conditions. The results of this study not only confirm the theoretical robustness of the method but also highlight its practical utility across a broad spectrum of function types, including smooth quadratic, transcendental, and non-smooth functions with singularities. Through step-by-step numerical examples, the method's consistency, efficiency, and adaptability were demonstrated. In this section, we interpret the key findings in light of the original research questions, relate them to established optimization theory, and situate them within the broader scientific and engineering literature on one-dimensional optimization.

The empirical results strongly align with the original research objectives, demonstrating that the Golden Section Search (GSS) method is a reliable derivative-free optimization technique, particularly applicable in scenarios where derivative information is unavailable or unreliable. Across all test cases, the method consistently converged toward the minimum, thereby confirming the mathematical convergence properties analyzed in earlier sections. Furthermore, its performance across various function types—including quadratic, oscillatory, and non-differentiable forms—highlights the method's broad adaptability and underscores its relevance in engineering, computational, and scientific modeling contexts.

The Golden Section Search (GSS) method demonstrated consistent performance across three distinct optimization scenarios, each chosen to represent varying levels of analytical complexity. In the first example, a smooth and convex quadratic function $f(x)=3x^2+20x-1$ served as a classical benchmark. The method rapidly converged to the minimum within five iterations, effectively exploiting the golden ratio $k\approx 0.618$ to reduce the uncertainty interval geometrically. This confirmed the theoretical convergence properties of GSS, especially its exponential rate of interval reduction. Example 2, also involving a smooth unimodal function, reinforced the method's behavior by showing that early elimination of non-promising subintervals leads to efficient localization of the extremum. Notably, GSS retained previously evaluated points, reducing computational overhead and highlighting its practicality in resource-constrained settings.

Example 3 introduced a significantly more complex function, $f(x)=|\sin(x)\cdot\ln(x+2.5)|$, characterized by non-smoothness and a logarithmic singularity near $x=-2.5$. Despite lacking differentiability and violating the assumptions of classical methods, the GSS algorithm converged reliably to a global minimum $x=0$ after 13 iterations, entirely within the defined tolerance. This result illustrates the robustness of GSS in handling non-standard objective functions where derivative-based methods are inapplicable. Since GSS relies only on function evaluations and assumes unimodality—not continuity or differentiability—it is particularly well-suited for real-world scenarios involving noisy, empirical, or simulation-derived functions. These observations are consistent with findings from (Abd Elaziz et al., 2017), (Hashemi et al., 2022), and (Noroozi et al., 2022), who emphasized the method's flexibility under non-ideal conditions.

Taken together, the results from all three examples underscore the reliability, simplicity, and generalizability of the GSS method. It performs exceptionally well on smooth, convex functions with rapid convergence and maintains accuracy even in complex, non-differentiable cases. The variation in iteration counts—from 5 to 13—reflects not a limitation of the method but rather the intrinsic complexity of the functions involved. This confirms that GSS offers a robust trade-off between efficiency and general applicability, making it a powerful tool in scientific and engineering optimization problems, especially where gradient information is unavailable or unreliable.

The findings of this study align closely with the results reported by (Abd Elaziz et al., 2017) and (Malekian et al., 2019), both of whom emphasized the critical role of unimodality in the success of derivative-free optimization algorithms. While those prior studies offered valuable theoretical frameworks and simulation results, they primarily focused on high-level assessments of GSS behavior. In contrast, the present research contributes to the literature by offering a comprehensive, step-by-step procedural demonstration of the GSS method in action, backed by detailed analytical validation and applied numerical examples. This structured exposition addresses a noticeable gap in the literature regarding the practical implementation of the method across functions with varying characteristics—smooth, transcendental, and non-differentiable.

When compared to classical optimization methods such as Brent's Method and the Newton-Raphson technique, GSS exhibits a unique trade-off. While Newton-Raphson and Brent's algorithms typically converge faster when the objective function is twice differentiable and well-behaved, their reliance on first and second derivatives renders them less reliable or inapplicable in cases where derivatives are undefined, discontinuous, or noisy. This limitation is particularly evident in real-world problems involving simulation-based models, empirical measurements, or experimental data, where smoothness and differentiability cannot be guaranteed. In such scenarios, GSS demonstrates superior robustness, offering guaranteed convergence with minimal assumptions. This observation reinforces Bertsekas (2016), who noted the increasing need for adaptable, derivative-free methods in practical engineering applications, where modeling errors and irregularities are the norm rather than the exception.

Moreover, a distinctive contribution of this study lies in its attention to the geometric foundations of the Golden Section Search method, which are rooted in the classical concept of the golden ratio—a concept historically attributed to Leonardo da Vinci and the aesthetics of proportion. While this geometric basis is well-known in mathematics and the arts, it has rarely been emphasized or visualized within the optimization literature. By integrating graphical illustrations and geometric reasoning into the convergence process, this study enhances the interpretability of the algorithm for both learners and practitioners. This pedagogical value makes the research not only a technical contribution to the field of optimization but also a resource for teaching and communicating algorithmic intuition more effectively.

The findings have several implications for both theory and practice:

Mathematically, the study reaffirms the sufficiency of the first and second-order derivative tests, but also shows the need for higher-order conditions in ambiguous cases—demonstrating the complementary role of GSS in such contexts.

Practically, the method holds strong potential within mathematical modules, particularly in solving problems involving empirically defined functions, noisy numerical data, or discrete input values—such as those encountered in mathematical modeling, applied analysis, and instructional settings for numerical optimization.

The algorithm's low computational overhead, coupled with deterministic convergence, makes it ideal for embedded systems and real-time decision-making models where resources are limited.

Despite its numerous strengths and widespread applicability, the Golden Section Search (GSS) method has inherent limitations that must be carefully considered when selecting an optimization approach. Notably, these limitations affect its performance and suitability in certain problem domains, particularly those involving complex or non-ideal function characteristics. Key constraints include:

It is not optimal for multimodal functions, as it assumes a single minimum within the interval and may converge to a non-global extremum in the presence of multiple local minima due to the absence of global search mechanisms.

Its convergence speed, while consistent, may be outpaced by gradient-based or quasi-Newton methods when derivative information is available.

To address these gaps, future research could:

Extend the method to hybrid metaheuristics that switch between GSS and other methods based on function characteristics.

Develop multidimensional adaptations of GSS for problems in higher dimensions, possibly by embedding GSS within coordinate descent frameworks.

Investigate stochastic variants for optimization under uncertainty or noisy data, relevant to real-time and probabilistic modeling.

CONCLUSION

This study has comprehensively examined the Golden Section Search (GSS) method for one-dimensional function optimization, reaffirming its strong theoretical foundations and practical efficiency. Key contributions include the formalization of its mathematical framework, detailed stepwise algorithmic procedures, and validation through numerical examples involving both algebraic and transcendental functions.

As a derivative-free optimization technique, the GSS method demonstrates robust reliability, particularly for unimodal functions, with guaranteed convergence to a solution within predetermined accuracy thresholds. Its computational efficiency and applicability to non-differentiable functions render it an indispensable tool in various scientific and engineering domains.

From an applied perspective, this work advances current understanding by highlighting GSS's robustness across diverse function types, accompanied by explicit implementation guidelines and termination criteria. Such insights facilitate the broader adoption of GSS in scenarios where derivative information is either unavailable or unreliable.

Future research directions include extending the method to multimodal optimization through hybrid algorithms, adapting its principles for higher-dimensional problems, implementing adaptive stopping criteria to optimize computational cost versus accuracy dynamically, and applying the method to complex real-world problems, such as those encountered in machine learning and control engineering.

In summary, this research not only reinforces the Golden Section Search as a cornerstone of derivative-free optimization but also paves the way for its ongoing evolution and wider application, ensuring its sustained relevance in tackling advanced optimization challenges.

AUTHORS CONTRIBUTIONS

1. Hamidullah Noori conceptualized the study, designed the research framework, and implemented the Golden Section Search algorithm using both Mathematica and MATLAB.
2. Rahman Basharat conducted the mathematical analysis, verified the correctness of numerical computations, and contributed to drafting the methodology and discussion sections.
3. Mohammad Zarif Mehrzad performed the literature review, managed references using Zotero, and generated dynamic visualizations and diagrams in GeoGebra.

All authors contributed to writing the manuscript and provided critical revisions.

All authors reviewed and approved the final version of the manuscript.

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CONFLICT OF INTEREST STATEMENT

The authors declare that there is no conflict of interest.

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