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Generalized Formulas for Summation and Alternating Summation of Jacobsthal and Jacobsthal – Lucas Numbers

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ABSTRACT

Among integer sequences, the Jacobsthal and Jacobsthal-Lucas sequences are the most significant, with applications in many areas of mathematics and the applied sciences. Recently, there has been a lot of research on these numbers, their important properties, and particular summation formulas for these sequences; there are also a few generalizations in the literature. In the present paper, motivated by recent works on these number sequences, generalized summation and alternating summation formulas are derived. In particular, generalized summation formulas for the Jacobsthal and Jacobsthal–Lucas numbers are first established. that is, the summation of the form $\sum_{k=1}^{n} J_{mk+r}$ and $\sum_{k=1}^{n} j_{mk+r}$, where J_n and j_n are the nth Jacobsthal and Jacobsthal – Lucas numbers, respectively, and m and r are any integers such that $m \neq 0$. Furthermore, generalized summation formulas with alternating signs are obtained, that is, the summation of the form of $\sum_{k=1}^{n} (-1)^{k-1} J_{mk+r}$ and $\sum_{k=1}^{n} (-1)^{k-1} j_{mk+r}$, where m and r are any integers such that $m \neq 0$. For special values of m and r, the corresponding particular cases are given for every generalized case. One of the other key findings of this paper is the summation and alternating summation formulas for the Jacobsthal and Jacobsthal-Lucas numbers with negative indices; that is, the summation of the form of $\sum_{k=1}^{n} J_{-k}$, $\sum_{k=1}^{n} j_{-k}$, $\sum_{k=1}^{n} (-1)^{k-1} J_{-k}$, $\sum_{k=1}^{n} (-1)^{k-1} j_{-k}$ and some more are obtained.

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INTRODUCTION

In almost every aspect of modern mathematics and science, integer sequences are used. The generalized Horadam numbers, first introduced by Horadam in (Horadam, 1965), are defined by the following recurrence relation

$$w_n = pw_{n-1} + q\, w_{n-2}, \quad n \geq 2,$$

with initial values of $w_0 = a$, $w_1 = b$, such that p, q, a and b are integers. Particular cases of Horadam numbers are Fibonacci numbers ($p = 1$, $q = 1$, $a = 0$, $b = 1$), Lucas numbers ($p =$

1, $q = 1$, $a = 2$, $b = 1$), Pell numbers ($p = 2$, $q = 1$, $a = 0$, $b = 1$), Pell – Lucas numbers ($p = 2$, $q = 1$, $a = 2$, $b = 2$), Jacobsthal numbers ($p = 1$, $q = 2$, $a = 0$, $b = 1$) and Jacobsthal – Lucas numbers ($p = 1$, $q = 2$, $a = 2$, $b = 1$). Few of these numbers sequences which are used throughout this paper are as follows.

The Fibonacci number sequence, defined by the following recurrence relation, is the most significant integer sequence that has been thoroughly studied from both an algebraic and combinatorial standpoint,

$$F_n = F_{n-1} + F_{n-2}, n \geq 2,$$

with initial conditions of $F_0 = 0$ and $F_1 = 1$, that is, every number in this sequence is the sum of the previous two numbers. Another number sequence, the Lucas sequence, is defined by the same recurrence relation as the previous one, but with different initial values of $L_0 = 2$ and $L_1 = 1$ (Koshy, 2001). These sequences are listed at A000045 and A000032 in (Sloane, 1964), respectively; the first few Fibonacci numbers are 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, ... and Lucas numbers are 2, 1, 3, 4, 7, 11, 18, 29... (Koshy, 2001). For additional information about this and some more sequence, the reader can consult (Bicknell, 1975; Brousseau, 1968; Horadam, 1961; Horadam, 1961; Koshy, 2001; Mansoor Kakar & Mehrad, 2025; Melham, 1999; Silvester, 1979; Subba Rao, 1935).

Jacobsthal and Jacobsthal-Lucas number sequences are another sequence of integers that is just as significant as the Fibonacci and Lucas sequences. These sequences have applications in number theory, combinatorics, coding theory, and even electrical engineering. The second-order recurrence relations listed below define the Jacobsthal and Jacobsthal-Lucas numbers for $n \geq 2$.

$$J_n = J_{n-1} + 2\,J_{n-2}, J_0 = 0, J_1 = 1,$$

and

$$j_n = j_{n-1} + 2\,j_{n-2}, j_0 = 2, j_1 = 1,$$

respectively. These sequences are listed at A001045 and A014551 in (Sloane, 1964), the characteristic equation of these number sequences is $x^2 - x - 2 = 0$, with roots of $\alpha = 2$ and $\beta = -1$. The first few Jacobsthal and Jacobsthal-Lucas numbers are listed in Table 1 (Horadam, 1996).

***Table 1**. First few Jacobsthal and Jacobsthal-Lucas numbers*

n	0	1	2	3	4	5	6	7	8	9	10
J_n	0	1	1	3	5	11	21	43	85	171	341
j_n	2	1	5	7	17	31	65	127	257	511	1025

Binet's formulas for Jacobsthal and Jacobsthal-Lucas sequences are as follows (Horadam, 1997).

$$J_n = \frac{2^n - (-1)^n}{3}, \tag{1. 4}$$

and

$$j_n = 2^n + (-1)^n \, . \tag{1. 5}$$

For more details and relations regarding these number sequences, see (Babadag et al., 2024; Catarino et al., 2015; Horadam, 1988; Horadam, 1996; Horadam, 1997; Uygun, 2019).

The generating functions for Jacobsthal and Jacobsthal-Lucas sequences are as follows:

$$F(x) = \sum_{k=0}^{\infty} J_k x^k = \frac{x}{1-x-2x^2},$$

and

$$G(x) = \sum_{k=0}^{\infty} j_k x^k = \frac{2-x}{1-x-2x^2},$$

respectively (Dasdemir, 2019). For more about generating functions, see (Garaham, 1989) and (Wilf, 2006). In (Hoggatt, 1978), the author obtained the following explicit formulas for J_n and j_n,

$$J_n = \sum_{i=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \binom{n-i-1}{i} 2^i \, , \tag{1. 6}$$

and

$$j_n = \sum_{i=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \binom{n-i}{i} \frac{n}{n-i} 2^i \, , \tag{1. 7}$$

respectively, where $\lfloor n \rfloor$ is floor function or greatest integer function, which is defined for a real number x as follows

$$\lfloor x \rfloor = \text{the largest integer } n \text{ such that } n \le x \, . \tag{1. 8}$$

Similarly, the ceiling function denoted by $\lceil x \rceil$ is defined as follows

$$\lceil x \rceil = \text{the smallest integer } n \text{ such that } n \ge x \, . \tag{1. 9}$$

These functions have the following important properties which are used throughout this paper

$$\lfloor n \rfloor = \lceil n \rceil = n \, , n \in \mathbb{Z}$$

$$\lceil x + n \rceil = \lceil x \rceil + n$$

$$\lfloor x + n \rfloor = \lfloor x \rfloor + n$$

$$\left\lfloor \frac{n}{2} \right\rfloor + \left\lceil \frac{n}{2} \right\rceil = n$$

$$\left\lceil \frac{n}{2} \right\rceil - \left\lfloor \frac{n}{2} \right\rfloor = \begin{cases} 1, & n \text{ is odd} \\ 0, & n \text{ is even} \end{cases},$$

where n is an integer (Garaham, 1989). The n^{th} Jacobsthal and Jacobsthal–Lucas numbers with negative indices are as

$$J_{-n} = (-1)^{n+1} \frac{J_n}{2^n}, \tag{1. 10}$$

and

$$j_{-n} = (-1)^{n} \frac{j_n}{2^n}, \tag{1. 11}$$

respectively, the following relations hold

$$2^n \left(J_{-n} + J_n\right) = 3\, J_n^2,$$

$$2^n \left(j_{-n} + j_n\right) = j_n^2.$$

A few Jacobsthal and Jacobsthal-Lucas numbers with negative indices are listed in Table 2 (Dasdemir, 2019).

***Table 2**. Few Jacobsthal and Jacobsthal-Lucas numbers with negative indices*

n	0	1	2	3	4	5	6	7	8	9	10
J_{-n}	0	$\frac{1}{2}$	$-\frac{1}{4}$	$\frac{3}{8}$	$-\frac{5}{16}$	$\frac{11}{32}$	$-\frac{21}{64}$	$\frac{43}{128}$	$-\frac{85}{256}$	$\frac{171}{512}$	$-\frac{341}{2024}$
j_{-n}	2	$-\frac{1}{2}$	$\frac{5}{4}$	$-\frac{7}{8}$	$\frac{17}{16}$	$-\frac{31}{32}$	$\frac{65}{64}$	$-\frac{127}{128}$	$\frac{257}{256}$	$-\frac{511}{512}$	$\frac{1025}{1024}$

The following interrelations are hold (Horadam, 1996)

$j_n J_n = J_{2n}$,

$j_n = J_{n+1} + 2\, J_{n-1}$,

$9\, J_n = j_{n+1} + 2\, j_{n-1}$,

$j_{n+1} + j_n = 3\,(J_{n+1} + J_n) = 3 \cdot 2^n$,

$j_{n+1} - 2\, j_n = 3\,(2\, J_n - J_{n+1}) = 3\,(-1)^{n+1}$,

$j_{n+r} + j_{n-r} = 3\,(J_{n+r} + J_{n-r}) + 4\,(-1)^{n-r} = 2^{n-r}\,(2^{2r} + 1) + 2\,(-1)^{n-r}$,

$j_{n+r} - j_{n-r} = 3\,(J_{n+r} - J_{n-r}) = 2^{n-r}\,(2^{2r} - 1)$,

$j_n = 3\, J_n + 2\,(-1)^n$,

$3\, J_n + j_n = 2^{n+1}$,

$J_n + j_n = 2\, J_{n+1}$.

As mathematicians are always interested in generalizing topics and concepts across every area of mathematics, so are the Jacobsthal–Lucas number sequences. One of these generalizations is the k-Jacobsthal number sequences, defined in (Jhala et al., 2013) for $n \geq 1$ as follows

$$J_{k,n+1} = k\,J_{k,n} \ + 2\ J_{k,n-1}\,, J_{k,0} = 0, J_{k,1} = 1.$$

For $k = 1$, the usual Jacobsthal numbers have been obtained. Some properties and identities of these numbers are given in (Jhala et al., 2014), and also the following summation formula is obtained

$$\sum_{i=0}^{n} \ J_{k,ai+r} = \frac{-J_{k,r}-(-2)^r J_{k,a-r}}{J_{k,a+1}+2J_{k,a-1}-(-2)^a-1},$$

where a and r are integers. Similarly, the k-Jacobsthal – Lucas numbers are defined as follows (Campos et al., 2014)

$$j_{k,n+1} = k\,j_{k,n} \ + 2\ j_{k,n-1}\,, j_{k,0} = 2, j_{k,1} = k.$$

In (Bród & Michalski, 2022), the authors defined another generalization of the Jacobsthal–Lucas numbers as follows.

$$J(\,k,n\,) = (\,k - 1\,)\,J(\,k,n-1\,) + k\ J(\,k,n-2\,),\ \ n \geq 2,$$

with initial values of $J(\,k,0\,) = 0$ and $J(\,k,1\,) = 1$, and

$$j(\,k,n\,) = (\,k - 1\,)\,j(\,k,n-1\,) + k\ j(\,k,n-2\,),\ \ n \geq 2,$$

with initial values of $j(\,k,0\,) = 2$ and $j(\,k,1\,) = 1$, where n and $k \geq 2$ are integers. They named these numbers the generalized Jacobsthal and generalized Jacobsthal–Lucas number sequences, respectively. For $k = 2$, the Jacobsthal and Jacobsthal–Lucas numbers are obtained, that is, $(\,2,n\,) = J_n$ and $j(\,2,n\,) = j_n$. The authors obtained in Theorem 3.6 the following summation formulas for these generalized sequences.

$$\sum_{i=0}^{n} J(\,k,i\,) = \frac{1}{2k-2}\,(J(\,k,n+1) + k\,J(\,k,n)\ - 1\,),$$

and

$$\sum_{i=0}^{n} j(\,k,i\,) = \frac{1}{2k-2}\,(j(\,k,n+1) + k\,j(\,k,n)\ + 2k + 5\,)\,.$$

And the following Corollary

$$\sum_{i=0}^{n} J_i = \frac{J_{n+2}-1}{2}\,, \tag{1. 12}$$

and

$$\sum_{i=0}^{n} j_i = \frac{j_{n+2}-1}{2}. \tag{1. 13}$$

In (Horadam, 1996), A. F. Horadam found the following summation formulas

$$\sum_{i=2}^{n} J_i = \frac{J_{n+2}-3}{2} , \tag{1. 14}$$

and

$$\sum_{i=2}^{n} j_i = \frac{j_{n+2}-3}{2} . \tag{1. 15}$$

There is no research considering the generalized summations, summations with alternative signs, and summation of Jacobsthal and Jacobsthal–Lucas numbers with negative indices, so the aim of the present paper is that, by using techniques in (Frontczak, 2018) this summations will be generalized and obtain the generalized summation formulas, that is, the summation of the form $\sum_{k=1}^{n} J_{mk+r}$ and $\sum_{k=1}^{n} j_{mk+r}$ with any integers $m \neq 0$ and r. Furthermore, It has been obtained the generalized summation formulas with alternating signs, that is, the summation of the form $\sum_{i=1}^{n} (-1)^{k-1} J_{mk+r}$ and $\sum_{i=1}^{n} (-1)^{k-1} j_{mk+r}$ with integers $m \neq 0$ and r. By using negative values of m, it will obtain the summation and alternating summation formulas for Jacobsthal and Jacobsthal-Lucas numbers with negative indices.

So, the purpose of the present paper is to obtain the generalized summation and alternating summation formulas, and the main questions of this research are as follows:

1. Can generalized summation formulas for these numbers be found, from which all summation results can be derived?
2. How may generalized summation formulas with alternating signs be obtained?
3. Can generalized summation and alternating summation formulas for these numbers with negative indices be derived?

METHODS AND MATERIALS

This research focuses on the summation and alternating summation of Jacobsthal and Jacobsthal-Lucas number sequences. In many articles such as (Horadam, 1996; Jhala et al.,2013; Jhala et al., 2014) there are some summation formulas for these numbers sequences, but these summations are some special cases and there are no papers and researches considering the alternating summation formulas for these sequences.

Therefore, an interesting research gap has been identified: obtaining generalized formulas for the summation and alternating summation of Jacobsthal and Jacobsthal-Lucas numbers. That is, all results given in (Horadam, 1996; Jhala et al.,2013; Jhala et al., 2014) and some other papers will be generalized. By using the techniques in (Frontczak, 2018) and other mathematical techniques used for integer sequences in (Garaham et al., 1989), the results of this article have been obtained.

After obtaining the generalized summation and alternating summation formulas, special values were substituted to derive particular cases of these formulas appearing in the literature.

The methodology of this study is as follows. First, two lemmas from (Frontczak, 2018), are established and used as the foundation for the subsequent results, these Lemmas have some techniques to find these summations, and a Lemma which involves the important relation of these number sequences. Next, Theorems 1 and 2 establish generalized summation formulas for the Jacobsthal and Jacobsthal–Lucas sequences. Thereafter, by assigning suitable special values to the parameters, Corollaries 1–4 provide particular summation formulas for these sequences with positive and negative indices, recovering several known results as special cases.

Next, in Theorems 3 and 4, the alternating summation formulas for Jacobsthal and Jacobsthal-Lucas numbers have obtained, and there corresponding corollaries deal with some special cases and also the alternating summations for these numbers with negative indices.

FINDINGS

In this section, generalized summation formulas and generalized alternating summation formulas are derived, and several particular cases are discussed through corollaries. The discussion begins with the following lemmas.

Lemma 1. Let $f(k)$ be a real sequence, m, n, and p be positive integers, then

$$\sum_{k=1}^{n}\Big(f\big(m(k+p)\big)-f\big(m(k-p)\big)\Big)=\sum_{k=n+1-p}^{n+p} f(mk)-\sum_{k=1-p}^{p} f(mk)$$

Lemma 2. Let $f\ (k)$ be a real sequence and m, n, and p be positive integers; then

$$\sum_{k=1}^{n}(-1)^{k-1}\Big(f\big(m(k+p)\big)-f\big(m(k-p)\big)\Big)$$

$$=\sum_{k=n+1-p}^{n+p}(-1)^{k+p-1}f(mk)-\sum_{k=1-p}^{p}(-1)^{k+p-1}f(mk)$$

Proof: For proofs of these lemmas and more information, see (Frontczak, 2018).

Lemma 3. For Jacobsthal and Jacobsthal–Lucas numbers, the following identities hold

$$J_{k+n}-J_{k-n}=\frac{J_{2n}}{2^n}\left(3J_k+(-1)^k\right) \quad (2.1)$$

$$j_{k+n}-j_{k-n}=\frac{J_{2n}}{2^n}\left(j_k-(-1)^k\right) \quad (2.2)$$

Proof. Using Equality (1. 4), it follows that

$$J_{k+n}-J_{k-n}=\frac{1}{3}\left(2^{k+n}-(-1)^{k+n}\right)-\frac{1}{3}\left(2^{k-n}-(-1)^{k-n}\right)$$

$$= \frac{1}{3} 2^k \left(2^n - 2^{-n} \right)$$

$$= \frac{1}{3} 2^k \left(2^n - (-1)^k + (-1)^k \right) \frac{3J_{2n}}{2^n}$$

$$= \frac{J_{2n}}{2^n} \left(3 J_k + (-1)^k \right).$$

On the other hand by using Equality (1.5),

$$j_{k+n} - j_{k-n} = \left(2^{k+n} + (-1)^{k+n} \right) - \left(2^{k-n} + (-1)^{k-n} \right)$$

$$= 2^k \left(2^n - 2^{-n} \right)$$

$$= \frac{3J_{2n}}{2^n} \left(2^k - (-1)^k + (-1)^k \right)$$

$$= \frac{3J_{2n}}{2^n} \left(j_k - (-1)^k \right)$$

Theorem 1 (Sum of Jacobsthal numbers). For integers m and r with $m \neq 0$, the generalized sum of Jacobsthal numbers is given by

$$\sum_{k=1}^{n} J_{mk+r} = \frac{2^m}{3J_{2m}} \left((1 + 2^m)(J_{mn+r} - J_r) + (-1)^r J_m ((-1)^{mn} - 1) \right) - \frac{(-1)^r}{3} \left(\left\lceil \frac{n}{2} \right\rceil (-1)^m + \left\lfloor \frac{n}{2} \right\rfloor \right),$$

where $\lceil n \rceil$ and $\lfloor n \rfloor$ are ceiling and floor functions, respectively.

Proof. By using $f\ (k) = J_{k+r}$ and $p = 1$ in Lemma (1), it follows that

$$\sum_{k=1}^{n} \left(J_{m(k+1)+r} - J_{m(k-1)+r} \right) = \sum_{k=n}^{n+1} J_{mk+r} - \sum_{k=0}^{1} J_{mk+r}$$
$$= J_{mn+r} + J_{m(n+1)+r} - J_r - J_{m+r}.$$

By using Equality (1. 4) the last expression can be given as

$$\sum_{k=1}^{n} \left(J_{m(k+1)+r} - J_{m(k-1)+r} \right) = (1 + 2^m)(J_{mn+r} - J_r) + (-1)^r J_m ((-1)^{mn} - 1). \quad (2.3)$$

From Equality (2. 1) it concludes that

$$J_{k+n} - J_{k-n} = \frac{J_{2n}}{2^n} \left(3 J_k + (-1)^k \right),$$

by substituting n by m and k by $mk + r$, consequently

$$J_{m(k+1)+r} - J_{m(k-1)+r} = \frac{J_{2m}}{2^m} \left(3 J_{mk+r} + (-1)^{mk+r} \right),$$

by taking the summation, it follows that

$$\sum_{k=1}^{m} \left(J_{m(k+1)+r} - J_{m(k-1)+r}\right) = \sum_{k=1}^{n} \frac{J_{2m}}{2^m}(3J_{mk+r} + (-1)^{mk+r})$$

$$= \frac{J_{2m}}{2^m}\left(3\sum_{k=1}^{n} J_{mk+r} + \sum_{k=1}^{n} (-1)^{mk+r}\right). \quad (2.4)$$

Therefore

$$\sum_{k=1}^{n} (-1)^{mk+r} = (-1)^r \left(\left\lceil\frac{n}{2}\right\rceil (-1)^m + \left\lfloor\frac{n}{2}\right\rfloor\right). \quad (2.5)$$

Then from Equations (2. 3) and (2. 4), it can be conclude that

$$\frac{J_{2m}}{2^m}\left(3\sum_{k=1}^{n} J_{mk+r} + \sum_{k=1}^{n} (-1)^{mk+r}\right) = (1+2^m)(J_{mn+r} - J_r) + (-1)^r J_m((-1)^{mn} - 1).$$

Therefore by using Equations (2. 5), the result obtained as follows.

$$\sum_{k=1}^{n} J_{mk+r} = \frac{2^m}{3J_{2m}}\left((1+2^m)(J_{mn+r} - J_r) + (-1)^r J_m((-1)^{mn} - 1)\right) - \frac{(-1)^r}{3}\left(\left\lceil\frac{n}{2}\right\rceil (-1)^m + \left\lfloor\frac{n}{2}\right\rfloor\right)$$

Corollary 1. For the values of $m =$ 1, 2, 3 and $r =$ 0, 1, the following particular cases can be obtained easily,

$$\sum_{k=1}^{n} J_k = \begin{cases} 2J_n & ; n \text{ is even} \\ 2J_n - 1; & n \text{ is odd} \end{cases}$$

$$\sum_{k=1}^{n} J_{2k} = \frac{1}{3}(4J_{2n} - n)$$

$$\sum_{k=1}^{n} J_{2k+1} = \frac{1}{3}(4J_{2n+1} + n - 4)$$

$$\sum_{k=1}^{n} J_{3k} = \begin{cases} \frac{8}{7}J_{3n} & ; n \text{ is even} \\ \frac{1}{7}(8J_{3n} - 3) & ; n \text{ is odd} \end{cases}$$

Corollary 2. For negative values of m, the following summation formulas of Jacobsthal numbers with negative indices can be obtained:

$$\sum_{k=1}^{n} J_{-k} = \begin{cases} -J_{-n} + 1, & n \text{ is odd} \\ -J_{-n}, & n \text{ is even} \end{cases}$$

$$\sum_{k=1}^{n} J_{-2k} = -\frac{1}{3}(J_{-2n} + n)$$

$$\sum_{k=1}^{n} J_{-2k+1} = -\frac{1}{3}(J_{-2n+1} - n - 1)$$

$$\sum_{k=1}^{n} J_{-3k} = \begin{cases} -\frac{1}{7}(J_{-3n} - 3), & n \text{ is odd} \\ -\frac{1}{7}J_{-3n}, & n \text{ is even} \end{cases}$$

Theorem 2 (Sum of Jacobsthal-Lucas numbers). For integers m and r with $m \neq 0$, the sum of Jacobsthal–Lucas numbers is given by

$$\sum_{k=1}^{n} j_{mk+r} = \frac{2^m}{3J_{2m}}\left((1+2^m)(j_{mn+r} - j_r) + 3(-1)^r J_m(1-(-1)^{mn})\right) + (-1)^r\left(\left\lceil\frac{n}{2}\right\rceil(-1)^m + \left\lfloor\frac{n}{2}\right\rfloor\right)$$

Proof. By using $f(k) = j_{k+r}$ and $p = 1$ in Lemma (1), it follows that

$$\sum_{k=1}^{n} \left(j_{m(k+1)+r} - j_{m(k-1)+r}\right) = \sum_{k=n}^{n+1} j_{mk+r} - \sum_{k=0}^{1} j_{mk+r} = j_{mn+r} + j_{m(n+1)+r} - j_r - j_{m+r}.$$

By using Equation (1. 5) the last expression can be given as

$$\sum_{k=1}^{n} \left(j_{m(k+1)+r} - j_{m(k-1)+r}\right) = (1+2^m)(j_{mn+r} - j_r) + 3(-1)^r J_m(1-(-1)^{mn}). \quad (2.6)$$

From Equation (2. 2)

$$j_{k+n} - j_{k-n} = \frac{3J_{2n}}{2^n}(j_k - (-1)^k)$$

By substituting n by m and k by $mk + r$, it yields

$$j_{m(k+1)+r} - j_{m(k-1)+r} = \frac{3J_{2m}}{2^m}(j_{mk+r} - (-1)^{mk+r})$$

By taking the summation, it concludes that

$$\sum_{k=1}^{m} \left(j_{m(k+1)+r} - j_{m(k-1)+r}\right) = \sum_{k=1}^{n} \frac{3J_{2m}}{2^m}\left(j_{mk+r} - (-1)^{mk+r}\right)$$

$$= \frac{3J_{2m}}{2^m}\left(\sum_{k=1}^{n} j_{mk+r} - \sum_{k=1}^{n} (-1)^{mk+r}\right). \quad (2.7)$$

It follows that

$$\sum_{k=1}^{n} (-1)^{mk+r} = (-1)^r \left(\left\lceil\frac{n}{2}\right\rceil (-1)^m + \left\lfloor\frac{n}{2}\right\rfloor\right). \quad (2.8)$$

Then from Equations (2. 6) and (2. 7), it can be deduced that

$$\frac{3J_{2m}}{2^m}\left(\sum_{k=1}^{n} j_{mk+r} - \sum_{k=1}^{n} (-1)^{mk+r}\right) = (1+2^m)(j_{mn+r} - j_r) + 3(-1)^r J_m(1-(-1)^{mn}).$$

Finally, by using Equation (2. 8), It follows that

$$\sum_{k=1}^{n} j_{mk+r} = \frac{2^m}{3J_{2m}}\left((1+2^m)(j_{mn+r} - j_r) + 3(-1)^r J_m(1-(-1)^{mn})\right)$$
$$+(-1)^r \left(\left\lceil\frac{n}{2}\right\rceil (-1)^m + \left\lfloor\frac{n}{2}\right\rfloor\right)$$

Corollary 3. By using $m = 1, 2, 3$ and $r = 0, 1$, The following particular cases are obtained.

$$\sum_{k=1}^{n} j_k = \begin{cases} 2(j_n - 2); n \text{ is even} \\ 2j_n - 1; n \text{ is odd} \end{cases}$$

$$\sum_{k=1}^{n} j_{2k} = \frac{4}{3}(j_{2n} - 2) + n$$

$$\sum_{k=1}^{n} j_{2k+1} = \frac{4}{3}(j_{2n+1} - 1) - n$$

$$\sum_{k=1}^{n} j_{3k} = \begin{cases} \frac{8}{7}(j_{3n} - 2); n \text{ is even} \\ \frac{8}{7}j_{3n} - 1; n \text{ is odd} \end{cases}$$

Corollary 4. By using negative values of m, some summation formulas for the Jacobsthal–Lucas numbers with negative indices are obtained as follows.

$$\sum_{k=1}^{n} j_{-k} = \begin{cases} -j_{-n} - 1, & n \text{ is odd} \\ 2 - j_{-n}, & n \text{ is even} \end{cases}$$

$$\sum_{k=1}^{n} j_{-2k} = -\frac{1}{3}(j_{-2n} - 2) + n$$

$$\sum_{k=1}^{n} j_{-2k+1} = -\frac{1}{3}(j_{-2n+1} - 1) - n$$

$$\sum_{k=1}^{n} j_{-3k} = \begin{cases} -\frac{1}{7}j_{-3n} - 1, & n \text{ } is \text{ } odd \\ -\frac{1}{7}(j_{-3n} - 2), & n \text{ } is \text{ } even \end{cases}$$

Theorem 3 (Alternating sum of Jacobsthal numbers). For integers m and r with $m \neq 0$, the sum of Jacobsthal numbers with alternating signs is given by

$$\sum_{k=1}^{n} (-1)^{k-1} J_{mk+r} = \frac{2^m}{3J_{2m}}\Big((1-2^m)((-1)^n J_{mn+r} - J_r) + (-1)^r J_m\big(1-(-1)^{n(m+1)}\big)\Big) + \frac{(-1)^r}{3}\left(\left\lfloor\frac{n}{2}\right\rfloor - \left\lceil\frac{n}{2}\right\rceil(-1)^m\right)$$

Proof. By using $f\ (k) = J_{k+r}$ and $p = 1$ in Lemma (2), it follows that

$$\sum_{k=1}^{n} (-1)^{k-1}\big(J_{m(k+1)+r} - J_{m(k-1)+r}\big) = \sum_{k=n}^{n+1} (-1)^k J_{mk+r} - \sum_{k=0}^{1} (-1)^k J_{mk+r}$$
$$= (-1)^n J_{mn+r} + (-1)^{n+1} J_{m(n+1)+r} - J_r + J_{m+r}.$$

By using Equality (1. 4), the last expression can be given as

$$\sum_{k=1}^{n} (-1)^{k-1}\big(J_{m(k+1)+r} - J_{m(k-1)+r}\big) = (1-2^m)((-1)^n J_{mn+r} - J_r) + (-1)^r J_m\big(1-(-1)^{n(m+1)}\big) \quad (2.9)$$

From Equality (2. 1), it follows that

$$J_{k+n} - J_{k-n} = \frac{J_{2n}}{2^n}\left(3J_k + (-1)^k\right).$$

By substituting n by m and k by $mk + r$, it follows that

$$J_{m(k+1)+r} - J_{m(k-1)+r} = \frac{J_{2m}}{2^m}\left(3J_{mk+r} + (-1)^{mk+r}\right).$$

By taking the summation, this yields,

$$\sum_{k=1}^{m} (-1)^{k-1}\big(J_{m(k+1)+r} - J_{m(k-1)+r}\big) = \frac{J_{2m}}{2^m}\left(3\sum_{k=1}^{n} (-1)^{k-1} J_{mk+r} + \sum_{k=1}^{n} (-1)^{k-1}(-1)^{mk+r}\right) \quad (2.10)$$

One obtains that

$$\sum_{k=1}^{n} (-1)^{k(m+1)+r-1} = (-1)^{r-1}\left(\left\lfloor\frac{n}{2}\right\rfloor - \left\lceil\frac{n}{2}\right\rceil(-1)^m\right) \tag{2.11}$$

Then from Equalities (2. 9), (2. 10) and (2. 11), it follows that

$$\sum_{k=1}^{n} (-1)^{k-1}J_{mk+r} = \frac{2^m}{3J_{2m}}\Big((1-2^m)((-1)^n J_{mn+r} - J_r) + (-1)^r J_m\big(1-(-1)^{n(m+1)}\big)\Big) + \frac{(-1)^r}{3}\left(\left\lfloor\frac{n}{2}\right\rfloor - \left\lceil\frac{n}{2}\right\rceil(-1)^m\right)$$

Corollary 5. By using $m = 0, 1, 2$ and $r = 0, 1$, in the previous theorem, the following alternating summation formulas for the Jacobsthal numbers are obtained.

$$\sum_{k=1}^{n} (-1)^{k-1}J_k = \frac{2}{3}(-1)^{n+1}J_n + \frac{n}{3}$$

$$\sum_{k=1}^{n} (-1)^{k-1}J_{2k} = \begin{cases} -\frac{4}{5}J_{2n}, & n \text{ is even} \\ \frac{1}{5}(4J_{2n}+1), & n \text{ is odd} \end{cases}$$

$$\sum_{k=1}^{n} (-1)^{k-1}J_{2k+1} = \begin{cases} -\frac{4}{5}(J_{2n+1}-1), & n \text{ is even} \\ \frac{1}{5}(4J_{2n+1}+3), & n \text{ is odd} \end{cases}$$

$$\sum_{k=1}^{n} (-1)^{k-1}J_{3k} = \frac{8}{9}(-1)^{n+1}J_{3n} + \frac{n}{3}$$

Corollary 6. By using $m = -1, -2, -3$ and $r = 0, 1$, the following alternating summation formulas can be obtained easily for Jacobsthal numbers with negative indices

$$\sum_{k=1}^{n} (-1)^{k-1}J_{-k} = \frac{1}{3}(n - (-1)^n J_{-n})$$

$$\sum_{k=1}^{n} (-1)^{k-1}J_{-2k} = \begin{cases} \frac{1}{5}(J_{-2n}-1), & n \text{ is odd} \\ -\frac{1}{5}J_{-2n}, & n \text{ is even} \end{cases}$$

$$\sum_{k=1}^{n} (-1)^{k-1}J_{-2k+1} = \begin{cases} \frac{1}{5}(J_{-2n+1}+2), & n \text{ is odd} \\ -\frac{1}{5}(J_{-2n+1}-1), & n \text{ is even} \end{cases}$$

$$\sum_{k=1}^{n} (-1)^{k-1}J_{-3k} = \frac{1}{9}((-1)^{n-1}J_{-3n} + 3n)$$

Theorem 4 (Alternating sum of Jacobsthal-Lucas numbers). For integers m and r with $m \neq 0$, the sum of Jacobsthal–Lucas numbers with alternative signs is given by

$$\sum_{k=1}^{n} (-1)^{k-1} j_{mk+r} = \frac{2^m}{3J_{2m}} \Big((1-2^m)((-1)^n j_{mn+r} - j_r) - 3(-1)^r J_m \big(1-(-1)^{n(m+1)}\big) \Big) + (-1)^r \left(\left\lceil \frac{n}{2} \right\rceil (-1)^m - \left\lfloor \frac{n}{2} \right\rfloor \right)$$

Proof. By using $f(k) = j_{k+r}$ and $p = 1$ in Lemma (2), it yields

$$\sum_{k=1}^{n} (-1)^{k-1} \big(j_{m(k+1)+r} - j_{m(k-1)+r} \big) = (-1)^n j_{mn+r} + (-1)^{n+1} j_{m(n+1)+r} - j_r + j_{m+r}.$$

By using Equality (1. 5), the last expression can be given as

$$\sum_{k=1}^{n} (-1)^{k-1} \big(j_{m(k+1)+r} - j_{m(k-1)+r} \big) = (1-2^m)((-1)^n j_{mn+r} - j_r) - 3(-1)^r J_m \big(1-(-1)^{n(m+1)}\big) \qquad (2.12)$$

From Equality (2. 2), it follows that

$$j_{k+n} - j_{k-n} = \frac{3J_{2n}}{2^n} \left(j_k - (-1)^k \right).$$

By substituting n by m and k by $mk + r$, it yields

$$j_{m(k+1)+r} - j_{m(k-1)+r} = \frac{3J_{2m}}{2^m} \left(j_{mk+r} - (-1)^{mk+r} \right)$$

By taking the summation, the following result is obtained

$$\sum_{k=1}^{m} (-1)^{k-1} \big(j_{m(k+1)+r} - j_{m(k-1)+r} \big) = \frac{3J_{2m}}{2^m} \left(\sum_{k=1}^{n} (-1)^{k-1} j_{mk+r} - \sum_{k=1}^{n} (-1)^{k-1} (-1)^{mk+r} \right) \qquad (2.13)$$

One obtains that

$$\sum_{k=1}^{n} (-1)^{k(m+1)+r-1} = (-1)^{r-1} \left(\left\lfloor \frac{n}{2} \right\rfloor - \left\lceil \frac{n}{2} \right\rceil (-1)^m \right). \qquad (2.14)$$

Then from Equalities (2. 12), (2. 13) and (2. 14), it follows that

$$\sum_{k=1}^{n} (-1)^{k-1} j_{mk+r} = \frac{2^m}{3J_{2m}} \Big((1-2^m)((-1)^n j_{mn+r} - j_r) - 3(-1)^r J_m \big(1-(-1)^{n(m+1)}\big) \Big) + (-1)^r \left(\left\lceil \frac{n}{2} \right\rceil (-1)^m - \left\lfloor \frac{n}{2} \right\rfloor \right).$$

Corollary 7. For the values of $m = 1, 2, 3$ and $r = 0, 1$, the following special cases are obtained:

$$\sum_{k=1}^{n} (-1)^{k-1} j_k = \frac{2}{3}((-1)^{n+1} j_n + 2) - n$$

$$\sum_{k=1}^{n} (-1)^{k-1} j_{2k} = \begin{cases} -\frac{4}{5}(j_{2n} - 2); n \text{ is even} \\ \frac{4}{5} j_{2n} + 1; n \text{ is odd} \end{cases}$$

$$\sum_{k=1}^{n} (-1)^{k-1} j_{2k+1} = \begin{cases} -\frac{4}{5}(j_{2n+1} - 1); n \text{ is even} \\ \frac{4}{5}(j_{2n+1} + 3) - 1; n \text{ is odd} \end{cases}$$

$$\sum_{k=1}^{n} (-1)^{k-1} j_{3k} = \frac{8}{9}((-1)^{n+1} j_{3n} + 2) - n$$

Corollary 8. For the values of $m = -1, -2, -3$ and $r = 0, 1$, the following alternating summation formulas for Jacobsthal-Lucas numbers with negative indices are obtained:

$$\sum_{k=1}^{n} (-1)^{k-1} j_{-k} = -\frac{1}{3}((-1)^{n} j_{-n} + 3n - 2)$$

$$\sum_{k=1}^{n} (-1)^{k-1} j_{-2k} = \begin{cases} \frac{1}{5} j_{-2n} + 1, & n \; is \; odd \\ -\frac{1}{5} j_{-2n}, & n \; is \; even \end{cases}$$

$$\sum_{k=1}^{n} (-1)^{k-1} j_{-2k+1} = \begin{cases} \frac{1}{5}(j_{-2n+1} - 2), & n \; is \; odd \\ -\frac{1}{5}(j_{-2n+1} - 1), & n \; is \; even \end{cases}$$

$$\sum_{k=1}^{n} (-1)^{k-1} j_{-3k} = -\frac{1}{9}((-1)^{n} j_{-3n} - 2) - n$$

DISCUSSION

Jacobsthal and Jacobsthal–Lucas number sequences are the most significant sequences among the integer sequences, which are defined by second-order recurrences, and they are particular cases of generalized Horadam numbers.

Some papers discussed summation formulas for both Jacobsthal and Jacobsthal–Lucas numbers in a few special cases; however, there are no research papers that have considered the summation formulas of these numbers in a very general form. Also, so far, there are no papers that have considered the alternating summation formulas and summation formulas for these numbers with negative indices. This paper investigates new results for these integer sequences to address this problem.

The key findings of this research are as follows.

1. To answer the first research question, the generalized summation formulas for the Jacobsthal and Jacobsthal–Lucas numbers are obtained as follows:

$$\sum_{k=1}^{n} J_{mk+r} = \frac{2^m}{3J_{2m}} [(1 + 2^m) (J_{mn+r} - J_r) + (-1)^r J_m ((-1)^{mn} - 1)] - \frac{(-1)^r}{3} (\left\lceil \frac{n}{2} \right\rceil (-1)^m + \left\lfloor \frac{n}{2} \right\rfloor).$$

And

$$\sum_{k=1}^{n} j_{mk+r} = \frac{2^m}{3J_{2m}} [(1 + 2^m) (j_{mn+r} - j_r) + 3 (-1)^r J_m (1 - (-1)^{mn})] + (-1)^r (\left\lceil \frac{n}{2} \right\rceil (-1)^m + \left\lfloor \frac{n}{2} \right\rfloor).$$

Where J_n and j_n are the n^{th} Jacobsthal and Jacobsthal–Lucas numbers, respectively, m and r are integers with $m \neq 0$. This is the most generalized form for this summation, by assigning different values to $m$and r, several known identities previously established in the literature are recovered as special cases. Some of these results appear in (Horadam, 1996; Bród & Michalski, 2022; Jhala et al., 2013; Jhala et al., 2014).

2. One of the most significant results of this paper is obtaining the summation formulas for these numbers with negative indices. By using negative values of m in the above general formulas, the summations of the form of $\sum_{k=1}^{n} J_{-k}$, $\sum_{k=1}^{n} J_{-2k}$, $\sum_{k=1}^{n} J_{-2k+1}$, $\sum_{k=1}^{n} J_{-3k}$, $\sum_{k=1}^{n} j_{-k}$, $\sum_{k=1}^{n} j_{-2k}$, $\sum_{k=1}^{n} j_{-3k}$ and more are conclude. So far, no papers or references have considered these summations and obtained such results.

3. Another important result of this paper is obtaining the generalized alternating summation formulas for both sequences, that is, the summations of the following forms,

$$\sum_{k=1}^{n} (-1)^{k-1} J_{mk+r} = \frac{2^m}{3J_{2m}} [(1 - 2^m) ((-1)^n J_{mn+r} - J_r) + (-1)^r J_m (1 - (-1)^{n(m+1)})] + \frac{(-1)^r}{3} (\left\lfloor \frac{n}{2} \right\rfloor - \left\lceil \frac{n}{2} \right\rceil (-1)^m).$$

And

$$\sum_{k=1}^{n} (-1)^{k-1} j_{mk+r} = \frac{2^m}{3J_{2m}} [(1 - 2^m) ((-1)^n j_{mn+r} - j_r) + 3 (-1)^r J_m (1 - (-1)^{n(m+1)})] + (-1)^r (\left\lceil \frac{n}{2} \right\rceil (-1)^m - \left\lfloor \frac{n}{2} \right\rfloor).$$

Where m and r are integers with $m \neq 0$. By using particular values of m and r, the following particular cases are obtained. that is, the summations of the form of $\sum_{k=1}^{n} (-1)^{k-1} J_k$, $\sum_{k=1}^{n} (-1)^{k-1} J_{2k}$, $\sum_{k=1}^{n} (-1)^{k-1} J_{2k+1}$, $\sum_{k=1}^{n} (-1)^{k-1} J_{3k}$, $\sum_{k=1}^{n} (-1)^{k-1} j_k$, $\sum_{k=1}^{n} (-1)^{k-1} j_{2k}$ and more. The interesting thing is that there is no research involving this type of summation.

4. The last finding of this paper is obtaining the alternating summation of Jacobsthal and Jacobsthal-Lucas numbers with negative indices, if negative values of mmm are considered,, then the summations of the form of $\sum_{k=1}^{n} (-1)^{k-1} J_{-k}$, $\sum_{k=1}^{n} (-1)^{k-1} J_{-2k}$, $\sum_{k=1}^{n} (-1)^{k-1} J_{-2k+1}$, $\sum_{k=1}^{n} (-1)^{k-1} J_{-3k}$, $\sum_{k=1}^{n} (-1)^{k-1} j_{-k}$, $\sum_{k=1}^{n} (-1)^{k-1} j_{-2k}$ and more can be obtained; there are no papers that obtained such results.

For some particular values of m and r, the special cases of this summation that have been previously studied in the literature; for example, by using $m = 1$ and $r = 0$ in these generalized formulas, the following formulas are obtained in (Horadam, 1996; Bród & Michalski, 2022) with a slightly different form:

$$\sum_{k=1}^{n} J_k = \begin{cases} 2J_n & ; n \text{ is even} \\ 2J_n - 1; & n \text{ is odd} \end{cases} \quad \text{and} \quad \sum_{k=1}^{n} j_k = \begin{cases} 2(j_n - 2); & n \text{ is even} \\ 2j_n - 1; & n \text{ is odd} \end{cases}.$$

Also using these generalizations, the summations of $\sum_{k=1}^{n} J_{2k}$, $\sum_{k=1}^{n} J_{2k+1}$, $\sum_{k=1}^{n} J_{3k}$, $\sum_{k=1}^{n} j_{2k}$, $\sum_{k=1}^{n} j_{2k+1}$, $\sum_{k=1}^{n} j_{3k}$ and more are obtained by particular values of m and r.

Some papers for examples (Bród & Michalski, 2022; Campos et al., 2014; Horadam, 1996) in which the researchers obtained particular cases of these summations used Binet's formulas and geometric series, but in this paper, the new techniques for sequences are used.

In future research, researchers can use the same techniques to obtain generalized summation and alternating summation formulas for other integer sequences, such as Fibonacci, Lucas, Pell, Pell-Lucas, Padovan, and others. Furthermore, the researchers can generalize these results to third and higher-order recurrence relations.

CONCLUSION

In the present paper, some properties of special integer sequences, namely the Jacobsthal and Jacobsthal–Lucas number sequences, are studied. First, generalized summation formulas for both sequences are derived, along with corollaries describing special cases of these general results. Next, summation formulas with alternating signs for these integer sequences are investigated, and several particular cases are obtained. The most important of these special cases are the summation and alternating summation formulas for negative indices. Further researchers may obtain the same results for other integer sequences by using the techniques in this paper.

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