

Exploring n-Trigonometrically Convexity and Hermite-Hadamard Type Inequalities

Mohammad Mubarak¹, Abdul Wakil Baidar², Noorullah Noori³

¹Laghman University, Department of Mathematics, Faculty of Education, Laghman, Afghanistan

^{2,3}Kabul University, Department of Mathematics, Faculty of Mathematics, Kabul, Afghanistan

✉ E-mail: m.mubarak.94101@gmail.com (corresponding author)

ABSTRACT

In this paper, we introduce and systematically investigate n-trigonometrically convex functions, providing a rigorous definition and discussing their fundamental properties. We examine several algebraic and analytical properties of this newly defined class of functions in detail, revealing their structural behavior and relationships with existing convexity notions. To analyze this class, we use a methodological combination of trigonometric coefficients and polynomial structures. A key outcome of this work is derivation of Hermite–Hadamard-type inequalities specifically tailored to n-trigonometrically convex functions, inequalities that generalize the standard results already known for ordinary convex functions. In addition, a set of new integral inequalities is developed for functions whose first derivative, expressed in absolute value and raised to a power of one or higher, falls within the n-trigonometrically convex class. To establish these new analytical bounds, we use a rigorous methodology based on novel integral identities, which are evaluated with the aid of Hölder’s inequality together with the power-mean inequality. These inequalities provide refined bounds and generalizations that enrich the current theory of convex analysis. The obtained results not only broaden the scope of classical convexity concepts but also introduce new analytical tools that may be applied in different areas of mathematical analysis and inequality theory. In the final part of the paper, we demonstrate the applicability of the established results by applying them to well-known special means of real numbers, such as the arithmetic and geometric means. These applications confirm the practical value and usefulness of the proposed framework and highlight its potential for further developments in both theoretical and applied mathematics.

ARTICLE INFO

Article history:

Received: 6 April, 2026

Revised: 27 July, 2026

Accepted: 27 August, 2026

Published: 30 Sept, 2026

Keywords:

Convex function; Hermite-Hadamard inequality; n-Trigonometrically convex function; Special means

To cite this article: Mubarak, M., Baidar, A. W., Noori, N.(2026). Exploring n-Trigonometrically Convexity and Hermite-Hadamard Type Inequalities. *Journal of Natural Science Review*, 4(3), 1002-1018.
<https://doi.org/10.62810/jnsr.v4i3.474>

Link to this article: <https://kujnsr.com/JNSR/article/view/474>



Copyright © 2026 Author(s). This work is licensed under a Creative Commons Attribution-Noncommercial 4.0 International License

INTRODUCTION

Convexity is a foundational concept across numerous mathematical domains, particularly in mathematical analysis, geometry, and optimization theory. Foundational works in this field significantly expanded the analytical characterization of convex structures (Hadamard,

1893). Providing a versatile framework to describe the functional and set-theoretic properties, convexity plays a critical role in evaluating sharp upper and lower bounds in mathematical modeling (Dragomir & Agarwal, 1998). To address increasingly complex analytical challenges, classical convexity has undergone various extensions (Dragomir et al., 1995). A major milestone was the development of s -convex functions (Hudzik & Maligranda, 1994), which prompted subsequent researchers to formulate generalized convexity frameworks to broaden their analytical scope (Varošanec, 2007). Further progress led to enhanced theories of generalized convexity and associated integral bounds (Kadakal, 2018; İşcan, 2019). The emergence of n -polynomial convex functions established an effective path for refining integral inequalities (Toplu et al., 2020), a concept later expanded through n -polynomial exponentially s -convex functions (Tariq et al., 2021). More recently, Baidar and Kunt (2023) formulated the notion of GA- s -convex functions alongside corresponding Hermite–Hadamard type inequalities. In a related study, Baidar et al. (2023) defined a novel class of generalized harmonically convex functions, establishing associated integral bounds and highlighting their specialized instances.

Recent mathematical literature prominently features the integration of n -polynomial algebraic structures to establish sharper integral inequalities. For example, n -polynomial convex functions offered a fresh approach to refining the classical Hermite–Hadamard inequality (Toplu et al., 2020). This structural refinement was subsequently extended to n -polynomial exponentially s -convex functions, demonstrating strong applicability in deriving Ostrowski-type inequalities (Tariq et al., 2021). Further investigations broadened the operational range of generalized convexity across diverse analytical settings (Kunt et al., 2020; Baidar & Kunt, 2024). Additionally, various generalized classes such as P -functions (Alomari et al., 2012) and h -convex functions (Varošanec, 2007) have been extensively analyzed due to their significance in integral inequality theory.

Despite these developments, combining trigonometric convexity with n -polynomial frameworks remains an unaddressed research gap. Although trigonometrically convex functions are valuable tools in analytical contexts, standard formulations often lack the parametric flexibility required for refined estimations. Existing literature contains limited exploration of n -polynomial trigonometric structures. To bridge this gap, the current study introduces and formalizes the novel class of n -trigonometrically convex functions. The principal objective of this paper is to investigate the algebraic properties of this function class and to establish new extensions of the Hermite–Hadamard inequality. The derived results extend traditional trigonometric convexity while providing sharper analytical bounds applicable to broad mathematical disciplines.

The remainder of this paper is organized to present fundamental mathematical prerequisites before developing the core findings. Specifically, the primary objectives of this research are:

1. To introduce the framework of n -trigonometrically convex functions and analyze their core algebraic behavior.

2. To derive and evaluate Hermite–Hadamard type integral inequalities within this new setting.
3. To contrast the established n -trigonometric bounds and inequalities with existing formulations in the literature.

Definition 1. A mapping $g: J \rightarrow \mathbb{R}$ is said to be convex on the interval J if the relation

$$g(tu + (1 - t)v) \leq tg(u) + (1 - t)g(v),$$

holds for all $u, v \in J$ and $t \in [0, 1]$.

Definition 2. For any positive real parameters $m, n > 0$ the Euler Beta function, denoted as $\beta(m, n)$, is defined via the integral representation:

$$\beta(m, n) = \int_0^1 t^{m-1} (1 - t)^{n-1} dt.$$

Definition 3. A non-negative mapping $g: J \rightarrow \mathbb{R}$ belongs to the class of P -functions if the inequality

$$g(tx_1 + (1 - t)x_2) \leq g(x_1) + g(x_2),$$

holds for every pair $x_1, x_2 \in J$ and $t \in [0, 1]$. The collection of all P -functions defined over J is represented by $P(J)$ (Dragomir et al., 1995).

Definition 4. Let $h: J \rightarrow \mathbb{R}$ be a non-zero, non-negative function. A positive function $g: J \rightarrow \mathbb{R}$ is termed h -convex function, if

$$g(tv_1 + (1 - t)v_2) \leq h(t)g(v_1) + h(1 - t)g(v_2),$$

is satisfied for all $v_1, v_2 \in J$ and $t \in [0, 1]$. If the direction of the inequality sign is inverted, g is classified as an h -convex (Varošanec, 2007).

Definition 5. A nonnegative mapping $g: J \rightarrow \mathbb{R}$ is classified as trigonometrically convex on J if the relation:

$$g(tu_1 + (1 - t)u_2) \leq \left(\sin\left(\frac{\pi t}{2}\right)\right)g(u_1) + \left(\cos\left(\frac{\pi t}{2}\right)\right)g(u_2),$$

holds for all points $u_1, u_2 \in J$ and $t \in [0, 1]$ (Kadakal, 2018).

Definition 6. A nonnegative mapping $g: J \subset \mathbb{R} \rightarrow \mathbb{R}$ is said to belong to the class of n -polynomial convex function if

$$g(t\alpha + (1 - t)\beta) \leq \frac{1}{n} \cdot \sum_{i=1}^n [1 - (1 - t)^i] f(\alpha) + \frac{1}{n} \cdot \sum_{i=1}^n [1 - t^i] f(\beta),$$

holds for all $\alpha, \beta \in J$ and $t \in [0, 1]$, where $n \in \mathbb{N}$ (Toplu et al., 2020).

Definition 7. Let $n \in \mathbb{N}$ and $s \in [ln2, 4, 1]$. A non-negative function $g: I \subset \mathbb{R} \rightarrow \mathbb{R}$ is termed an n -polynomial exponentially s -convex function on I if

$$g(t\theta_1 + (1 - t)\theta_2) \leq \frac{1}{n} \cdot \sum_{i=1}^n (e^{st} - 1)^i g(\theta_1) + \frac{1}{n} \cdot \sum_{i=1}^n (e^{s(1-t)} - 1)^i g(\theta_2),$$

is satisfied for all $\theta_1, \theta_2 \in I$ and $t \in [0, 1]$ (Tariq et al., 2021).

Theorem 1. Let $m, n > 1$ be conjugate exponents satisfying $\frac{1}{m} + \frac{1}{n} = 1$. If $\gamma, \delta: [u, v] \rightarrow \mathbb{R}$ are integrable functions such that $|\gamma|^p$ and $|\delta|^q$ are integrable over $[u, v]$, then the inequality:

$$\int_u^v |\gamma(t) \delta(t)| dt \leq \left(\int_u^v |\gamma(t)|^m dt \right)^{\frac{1}{m}} \left(\int_u^v |\delta(t)|^n dt \right)^{\frac{1}{n}},$$

Holds (İşcan, 2019).

Theorem 2. Suppose that $J \subset \mathbb{R}$ is an interval and $g: J \rightarrow \mathbb{R}$ is a convex mapping. Then, for all $a_1, a_2 \in J$ with $a_1 < a_2$, the double inequality

$$g\left(\frac{a_1 + a_2}{2}\right) \leq \frac{1}{a_2 - a_1} \int_{a_1}^{a_2} g(x) dx \leq \frac{g(a_1) + g(a_2)}{2},$$

is satisfied (Hadamard, 1893).

Lemma 1. Assume that $g: J \subset \mathbb{R} \rightarrow \mathbb{R}$ is a differentiable function on the interior of J , where $\alpha_1, \alpha_2 \in J$ with $\alpha_1 < \alpha_2$. If $g' \in L[\alpha_1, \alpha_2]$, the identity

$$\frac{g(\alpha_1) + g(\alpha_2)}{2} - \frac{1}{\alpha_2 - \alpha_1} \int_{\alpha_1}^{\alpha_2} g(x) dx = \frac{\alpha_2 - \alpha_1}{2} \int_0^1 (1 - 2t) g'(t\alpha_1 + (1 - t)\alpha_2) dt,$$

Holds (Dragomir & Agarwal, 1998).

METHODS AND MATERIALS

This study used purely theoretical and analytical mathematical methods. The research focused on generalized notions of convexity for nonnegative real-valued functions defined on intervals of the real line. All results were derived using rigorous mathematical reasoning, algebraic manipulations, and analytical inequality techniques.

Research Design

This work was a theoretical study aimed at introducing a new class of functions, namely n -trigonometrically convex functions, and investigating their Basic algebraic and analytical properties. The study generalized existing concepts of convexity and examined the relationships between classical convexity, trigonometrically convex functions, and n -polynomial convexity.

Tools

The following mathematical tools were employed throughout the paper:

- Basic concepts of real analysis
- Definitions and properties of convex functions
- Algebraic inequalities
- Integral calculus
- Classical formulations of Hermite–Hadamard type inequalities

All definitions and known results were drawn from standard mathematical literature and generalized where necessary.

Methodological Framework

First, a new definition of n -trigonometrically convex functions was introduced. Based on this definition, several algebraic properties of this class were established. The proofs relied on direct analytical arguments, integral techniques, and inequality comparison methods.

New forms of Hermite–Hadamard type inequalities were then derived for the proposed class of functions. These inequalities were shown to refine or generalize the classical Hermite–Hadamard inequality as special cases. Furthermore, several particular cases were discussed and compared with existing results in the literature to demonstrate the generality and effectiveness of the new concept.

Data Sources

This study used no experimental data, numerical simulations, or statistical datasets. All results were obtained through theoretical analysis and mathematical proofs.

Ethical Considerations

Since this research was purely theoretical and did not involve human participants, animals, or sensitive data, ethical approval was not required.

FINDINGS

This section is devoted to introducing the novel class of n -trigonometrically convex functions. We present their core algebraic behavior and establish the corresponding Hermite–Hadamard-type inequalities associated with this formulation.

Definition 1. A non-negative mapping $g : J \rightarrow \mathbb{R}$ is said to be n -Trigonometrically convex function if the inequality

$$g(tu + (1-t)v) \leq \left(\frac{1}{n} \cdot \sum_{i=1}^n \left[\sin^i \left(\frac{\pi t}{2} \right) \right] \right) \cdot g(u) + \left(\frac{1}{n} \cdot \sum_{i=1}^n \left[\cos^i \left(\frac{\pi t}{2} \right) \right] \right) \cdot g(v),$$

holds for every $u, v \in J$ and $t \in [0, 1]$, where $n \in \mathbb{N}$.

Remark 1. Setting $n = 1$ in definition 1, the non-negative mapping $g : J \rightarrow \mathbb{R}$ reduces to the standard trigonometrically convex function (or 1-trigonometrically convex function), satisfying

$$g(tu + (1-t)v) \leq \left(\sin \left(\frac{\pi t}{2} \right) \right) g(u) + \left(\cos \left(\frac{\pi t}{2} \right) \right) g(v),$$

for all $u, v \in I$ and all $t \in [0, 1]$.

Remark 2. By defining the weight function

$$h(t) = \frac{1}{n} \cdot \sum_{i=1}^n \left[\sin^i \left(\frac{\pi t}{2} \right) \right],$$

it readily follows that any n -trigonometrically convex function falls within the framework of h -convex functions.

Remark 3. Every non-negative n -trigonometrically convex function g automatically constitutes a P -function. This is evident from the observation that,

$$g(tu + (1-t)v) \leq \left(\frac{1}{n} \cdot \sum_{i=1}^n \left[\sin^i \left(\frac{\pi t}{2} \right) \right] \right) g(u) + \left(\frac{1}{n} \cdot \sum_{i=1}^n \left[\cos^i \left(\frac{\pi t}{2} \right) \right] \right) g(v) \\ \leq g(u) + g(v),$$

for each $u, v \in I$ and $t \in [0, 1]$.

Example 1. Any constant mapping $g(x) = c \geq 0$ serves as an example of an n -trigonometrically convex function, which directly follows from the property that

$$\frac{1}{n} \cdot \sum_{i=1}^n \left[\sin^i \left(\frac{\pi t}{2} \right) \right] + \frac{1}{n} \cdot \sum_{i=1}^n \left[\cos^i \left(\frac{\pi t}{2} \right) \right] \geq 1,$$

holds for all $t \in [0, 1]$, then,

$$g(tu + (1-t)v) = c \leq \frac{1}{n} \cdot \sum_{i=1}^n \left[\sin^i \left(\frac{\pi t}{2} \right) + \cos^i \left(\frac{\pi t}{2} \right) \right] \cdot c \\ = \left(\frac{1}{n} \cdot \sum_{i=1}^n \left[\sin^i \left(\frac{\pi t}{2} \right) \right] \right) \cdot c + \left(\frac{1}{n} \cdot \sum_{i=1}^n \left[\cos^i \left(\frac{\pi t}{2} \right) \right] \right) \cdot c \\ = \left(\frac{1}{n} \cdot \sum_{i=1}^n \left[\sin^i \left(\frac{\pi t}{2} \right) \right] \right) \cdot g(u) + \left(\frac{1}{n} \cdot \sum_{i=1}^n \left[\cos^i \left(\frac{\pi t}{2} \right) \right] \right) \cdot g(v).$$

Theorem 1. Suppose $g : [a, b] \rightarrow \mathbb{R}$ belongs to the class of n -trigonometrically convex functions. For any non-negative constant $c \geq 0$, the scaled mapping cg is likewise n -trigonometrically convex.

Proof. Assume f is n -trigonometrically convex on $[a, b]$ and $c \geq 0$. Applying the definition for arbitrary $u, v \in [a, b]$ and all $t \in [0, 1]$, we obtain,

$$(cg)(tu + (1-t)v) \leq c \cdot \left(\frac{1}{n} \cdot \sum_{i=1}^n \left[\sin^i \left(\frac{\pi t}{2} \right) \right] g(u) + \frac{1}{n} \cdot \sum_{i=1}^n \left[\cos^i \left(\frac{\pi t}{2} \right) \right] g(v) \right) \\ = c \cdot \frac{1}{n} \cdot \sum_{i=1}^n \left[\sin^i \left(\frac{\pi t}{2} \right) \right] g(u) + c \cdot \frac{1}{n} \cdot \sum_{i=1}^n \left[\cos^i \left(\frac{\pi t}{2} \right) \right] g(v) \\ = \frac{1}{n} \cdot \sum_{i=1}^n \left[\sin^i \left(\frac{\pi t}{2} \right) \right] (cg)(u) + \frac{1}{n} \cdot \sum_{i=1}^n \left[\cos^i \left(\frac{\pi t}{2} \right) \right] (cg)(v).$$

■

Theorem 2. Let $g : J \rightarrow I$ be a convex mapping, and suppose $f : I \rightarrow \mathbb{R}$ is non-decreasing and n -trigonometrically convex. Then the composition function $f \circ g : J \rightarrow \mathbb{R}$ is itself n -trigonometrically convex.

Proof. To verify the n -trigonometric convexity of the composite mapping $f \circ g$, we invoke the classical convexity of g along with the monotonicity of f . For any choice of $u, v \in J$ and $t \in [0, 1]$, it follows that,

$$\begin{aligned}
 (f \circ g)(tu + (1-t)v) &= f(g(tu + (1-t)v)) \leq f(tg(u) + (1-t)g(v)) \\
 &\leq \frac{1}{n} \cdot \sum_{i=1}^n \left[\sin^i\left(\frac{\pi t}{2}\right) \right] f(g(u)) + \frac{1}{n} \cdot \sum_{i=1}^n \left[\cos^i\left(\frac{\pi t}{2}\right) \right] f(g(v)) \\
 &= \frac{1}{n} \cdot \sum_{i=1}^n \left[\sin^i\left(\frac{\pi t}{2}\right) \right] (f \circ g)(u) + \frac{1}{n} \cdot \sum_{i=1}^n \left[\cos^i\left(\frac{\pi t}{2}\right) \right] (f \circ g)(v).
 \end{aligned}$$

■

Theorem 3. Let $g : [a, b] \rightarrow \mathbb{R}$ be an integrable n -trigonometrically convex mapping with $a < b$. Then the integral inequality,

$$\frac{1}{b-a} \int_a^b g(u) du \leq \left[\frac{1}{n \cdot \pi} \sum_{i=1}^n \beta\left(\frac{i+1}{2}; \frac{1}{2}\right) \right] (g(a) + g(b)),$$

is satisfied.

Proof. Executing the variable change $u = ta + (1-t)b$ on the left-hand integral, we derive:

$$\begin{aligned}
 \frac{1}{b-a} \int_a^b g(u) du &\leq \int_0^1 g(ta + (1-t)b) dt \\
 &\leq \int_0^1 \left(\frac{1}{n} \cdot \sum_{i=1}^n \left[\sin^i\left(\frac{\pi t}{2}\right) \right] g(a) + \frac{1}{n} \cdot \sum_{i=1}^n \left[\cos^i\left(\frac{\pi t}{2}\right) \right] g(b) \right) dt \\
 &= \frac{g(a)}{n} \cdot \int_0^1 \left(\sum_{i=1}^n \left[\sin^i\left(\frac{\pi t}{2}\right) \right] \right) dt + \frac{g(b)}{n} \cdot \int_0^1 \left(\sum_{i=1}^n \left[\cos^i\left(\frac{\pi t}{2}\right) \right] \right) dt \\
 &= \frac{g(a)}{n} \cdot \sum_{i=1}^n \left(\int_0^1 \sin^i\left(\frac{\pi t}{2}\right) dt \right) + \frac{g(b)}{n} \cdot \sum_{i=1}^n \left(\int_0^1 \cos^i\left(\frac{\pi t}{2}\right) dt \right) \\
 &= \frac{2g(a)}{\pi \cdot n} \cdot \sum_{i=1}^n \left(\int_0^{\frac{\pi}{2}} \sin^i(x) dx \right) + \frac{2g(b)}{\pi \cdot n} \cdot \sum_{i=1}^n \left(\int_0^{\frac{\pi}{2}} \cos^i(x) dx \right) \\
 &= \frac{2g(a)}{\pi \cdot n} \cdot \sum_{i=1}^n \left(\frac{1}{2} \cdot \beta\left(\frac{i+1}{2}; \frac{1}{2}\right) \right) + \frac{2g(b)}{\pi \cdot n} \cdot \sum_{i=1}^n \left(\frac{1}{2} \cdot \beta\left(\frac{i+1}{2}; \frac{1}{2}\right) \right) \\
 &= \frac{1}{2} \cdot \frac{2}{\pi \cdot n} \cdot \sum_{i=1}^n \left(\beta\left(\frac{i+1}{2}; \frac{1}{2}\right) \right) \cdot (g(a) + g(b)) \\
 &= \left[\frac{1}{n \cdot \pi} \cdot \sum_{i=1}^n \beta\left(\frac{i+1}{2}; \frac{1}{2}\right) \right] \cdot (g(a) + g(b)).
 \end{aligned}$$

■

Remark 4. By choosing $n = 1$ in the inequality of Theorem 3, we obtain the reduced estimation:

$$\frac{1}{b-a} \int_a^b g(u) du \leq \frac{2}{\pi} \cdot (g(a) + g(b)).$$

which corresponds to the standard Hermite–Hadamard-type inequality for trigonometrically convex functions.

Theorem 4. Let $g : [a, b] \rightarrow \mathbb{R}$ be an n -trigonometrically convex mapping with $a < b$ and $g \in L[a, b]$. Then the following lower-bound inequality holds:

$$\frac{2 - \sqrt{2}}{2 - 2^{\frac{2-n}{2}}} \cdot \frac{n}{\sqrt{2}} \cdot g\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b g(x) dx.$$

Proof. Utilizing the definition of n -trigonometric convexity for the midpoint arguments, we write:

$$\begin{aligned} g\left(\frac{a+b}{2}\right) &= g\left(\frac{[ta + (1-t)b] + [(1-t)a + tb]}{2}\right) \\ &= g\left(\frac{1}{2} \cdot [ta + (1-t)b] + \frac{1}{2} \cdot [(1-t)a + tb]\right) \\ &\leq \frac{1}{n} \cdot \sum_{i=1}^n \left(\sin^i\left(\frac{\pi}{4}\right)\right) g(ta + (1-t)b) + \frac{1}{n} \cdot \sum_{i=1}^n \left(\cos^i\left(\frac{\pi}{4}\right)\right) g((1-t)a + tb) \\ &= \frac{1}{n} \cdot \sum_{i=1}^n \left(\frac{\sqrt{2}}{2}\right)^i g(ta + (1-t)b) + \frac{1}{n} \cdot \sum_{i=1}^n \left(\frac{\sqrt{2}}{2}\right)^i g((1-t)a + tb) \\ &= \frac{1}{n} \cdot \sum_{i=1}^n \left(\frac{\sqrt{2}}{2}\right)^i (g(ta + (1-t)b) + g((1-t)a + tb)), \end{aligned}$$

Integrating the resulting expression with respect to t over the interval $[0,1]$ yields,

$$\begin{aligned} \Rightarrow g\left(\frac{a+b}{2}\right) &\leq \frac{1}{n} \cdot \sum_{i=1}^n \left(\frac{\sqrt{2}}{2}\right)^i \left(\int_0^1 g(ta + (1-t)b) dt + \int_0^1 g((1-t)a + tb) dt \right) \\ &= \frac{1}{n} \cdot \frac{\sqrt{2}}{2} \cdot \frac{1 - \left(\frac{\sqrt{2}}{2}\right)^n}{1 - \frac{\sqrt{2}}{2}} \cdot \left(\frac{1}{b-a} \int_a^b g(x) dx + \frac{1}{b-a} \int_a^b g(y) dy \right) \\ &= \frac{\sqrt{2}}{n} \cdot \frac{1 - 2^{-\frac{1}{2} \cdot n}}{2 - \sqrt{2}} \cdot \frac{2}{b-a} \cdot \int_a^b g(x) dx \\ &= \frac{\sqrt{2}}{n} \cdot \frac{2 - 2^{\frac{2-n}{2}}}{2 - \sqrt{2}} \cdot \frac{1}{b-a} \int_a^b g(x) dx \\ &\Rightarrow \frac{2 - \sqrt{2}}{2 - 2^{\frac{2-n}{2}}} \cdot \frac{n}{\sqrt{2}} \cdot g\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b g(x) dx. \end{aligned}$$

■

Remark 5. Choosing $n = 1$ in Theorem 4 leads directly to the simplified inequality:

$$\frac{1}{\sqrt{2}} \cdot g\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b g(x) dx.$$

which corresponds to the lower-bound estimate for classical trigonometrically convex functions.

Next, we establish several novel integral bounds for functions whose first-order derivatives, when taken in absolute value and raised to powers $p \geq 1$, exhibit n -trigonometric convexity.

Theorem 5. Let $g : J \rightarrow \mathbb{R}$ be a continuously differentiable mapping on J with $a, b \in J$ ($a < b$) such that $g' \in L[a, b]$. If $|g'|$ is n -trigonometrically convex function on $[a, b]$, then:

$$\left| \frac{g(a) + g(b)}{2} - \frac{1}{b-a} \int_a^b g(x) dx \right| \leq \frac{b-a}{2n} \cdot \frac{|g'(a)| + |g'(b)|}{\sqrt{3\pi}} \cdot \sum_{i=1}^n \left(\beta\left(\frac{2i+1}{2}, \frac{1}{2}\right) \right)^{\frac{1}{2}}.$$

Proof. Applying Lemma 1 along with the fundamental properties of absolute values, we have:

$$\begin{aligned} & \left| \frac{g(a) + g(b)}{2} - \frac{1}{b-a} \int_a^b g(x) dx \right| \\ &= \left| \frac{b-a}{2} \cdot \int_0^1 (1-2t) \cdot g'(ta + (1-t)b) dt \right| \\ &\leq \frac{b-a}{2} \cdot \int_0^1 |(1-2t)| \cdot |g'(ta + (1-t)b)| dt \\ &\leq \frac{b-a}{2} \\ &\quad \cdot \int_0^1 |(1-2t)| \\ &\quad \cdot \left(\frac{1}{n} \cdot \sum_{i=1}^n \left[\sin^i\left(\frac{\pi t}{2}\right) \right] \cdot |g'(a)| + \frac{1}{n} \cdot \sum_{i=1}^n \left[\cos^i\left(\frac{\pi t}{2}\right) \right] \cdot |g'(b)| \right) dt \\ &= \frac{b-a}{2n} \cdot |g'(a)| \cdot \int_0^1 \sum_{i=1}^n |(1-2t)| \cdot \left[\sin^i\left(\frac{\pi t}{2}\right) \right] dt + \frac{b-a}{2n} \\ &\quad \cdot |g'(b)| \cdot \int_0^1 \sum_{i=1}^n |(1-2t)| \cdot \left[\cos^i\left(\frac{\pi t}{2}\right) \right] dt \\ &= \frac{b-a}{2n} \cdot |g'(a)| \cdot \sum_{i=1}^n \int_0^1 |(1-2t)| \cdot \left| \sin^i\left(\frac{\pi t}{2}\right) \right| dt + \frac{b-a}{2n} \\ &\quad \cdot |g'(b)| \cdot \sum_{i=1}^n \int_0^1 |(1-2t)| \cdot \left| \cos^i\left(\frac{\pi t}{2}\right) \right| dt \\ &\leq \frac{b-a}{2n} \cdot |g'(a)| \cdot \sum_{i=1}^n \left(\int_0^1 |(1-2t)|^2 dt \right)^{\frac{1}{2}} \left(\int_0^1 \left| \sin^i\left(\frac{\pi t}{2}\right) \right|^2 dt \right)^{\frac{1}{2}} \\ &\quad + \frac{b-a}{2n} \cdot |g'(b)| \cdot \sum_{i=1}^n \left(\int_0^1 |(1-2t)|^2 dt \right)^{\frac{1}{2}} \left(\int_0^1 \left| \cos^i\left(\frac{\pi t}{2}\right) \right|^2 dt \right)^{\frac{1}{2}} \end{aligned}$$

$$\begin{aligned}
&= \frac{b-a}{2n} \cdot \left(\int_0^1 |(1-2t)|^2 dt \right)^{\frac{1}{2}} \\
&\quad \cdot \left[|g'(a)| \cdot \sum_{i=1}^n \left(\frac{2}{\pi} \cdot \int_0^{\frac{\pi}{2}} \sin^{2i}(t) dt \right)^{\frac{1}{2}} \right. \\
&\quad \left. + |g'(b)| \cdot \sum_{i=1}^n \left(\frac{2}{\pi} \cdot \int_0^{\frac{\pi}{2}} \cos^{2i}(t) dt \right)^{\frac{1}{2}} \right] \\
&= \frac{b-a}{2n} \cdot \frac{1}{\sqrt{3}} \\
&\quad \cdot \left[|g'(a)| \cdot \sum_{i=1}^n \left(\frac{1}{\pi} \cdot \beta\left(\frac{2i+1}{2}, \frac{1}{2}\right) \right)^{\frac{1}{2}} + |g'(b)| \right. \\
&\quad \left. \cdot \sum_{i=1}^n \left(\frac{1}{\pi} \cdot \beta\left(\frac{2i+1}{2}, \frac{1}{2}\right) \right)^{\frac{1}{2}} \right] \\
&= \frac{b-a}{2n} \cdot \frac{1}{\sqrt{3}} \cdot \sum_{i=1}^n \left(\frac{1}{\pi} \cdot \beta\left(\frac{2i+1}{2}, \frac{1}{2}\right) \right)^{\frac{1}{2}} \cdot (|g'(a)| + |g'(b)|) \\
&= \frac{b-a}{2n} \cdot \frac{|g'(a)| + |g'(b)|}{\sqrt{3\pi}} \cdot \sum_{i=1}^n \left(\frac{1}{\pi} \cdot \beta\left(\frac{2i+1}{2}, \frac{1}{2}\right) \right)^{\frac{1}{2}}.
\end{aligned}$$

■

Theorem 6. Let $g : I \rightarrow \mathbb{R}$ be a differentiable mapping on I° with $a, b \in I$ ($a < b$) and $a, b \in I$, and suppose $g' \in L[a, b]$. If $|g'|^p$ is an n -trigonometrically convex function on the interval $[a, b]$, for $p > 1$, then

$$\begin{aligned}
&\left| \frac{g(a) + g(b)}{2} - \frac{1}{b-a} \int_a^b g(x) dx \right| \\
&\leq \frac{b-a}{2} \cdot \left(\frac{1}{1+q} \right)^{\frac{1}{q}} \cdot \left(\frac{|g'(a)|^p + |g'(b)|^p}{n \cdot \pi} \cdot \sum_{i=1}^n \beta\left(\frac{i+1}{2}, \frac{1}{2}\right) \right)^{\frac{1}{p}},
\end{aligned}$$

holds, where $q > 1$ satisfies $\frac{1}{q} + \frac{1}{p} = 1$.

Proof. Taking the absolute value in Lemma 1 and employing Hölder's inequality, it follows that:

$$\begin{aligned}
&\left| \frac{g(a) + g(b)}{2} - \frac{1}{b-a} \int_a^b g(x) dx \right| \\
&\leq \frac{b-a}{2} \int_0^1 |(1-2t)| \cdot |g'(ta + (1-t)b)| dt \\
&\leq \frac{b-a}{2} \cdot \left(\int_0^1 |(1-2t)|^q dt \right)^{\frac{1}{q}} \cdot \left(\int_0^1 |g'(ta + (1-t)b)|^p dt \right)^{\frac{1}{p}}
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{b-a}{2} \cdot \left(\frac{1}{1+q} \right)^{\frac{1}{q}} \\
&\quad \cdot \left(\int_0^1 \left[\frac{1}{n} \cdot \sum_{i=1}^n \left[\sin^i \left(\frac{\pi t}{2} \right) \right] \cdot |g'(a)|^p + \frac{1}{n} \cdot \sum_{i=1}^n \left[\cos^i \left(\frac{\pi t}{2} \right) \right] \right. \right. \\
&\quad \left. \left. \cdot |g'(b)|^p \right] dt \right)^{\frac{1}{p}} \\
&= \frac{b-a}{2} \cdot \left(\frac{1}{1+q} \right)^{\frac{1}{q}} \\
&\quad \cdot \left(\frac{|g'(a)|^p}{n} \cdot \sum_{i=1}^n \int_0^1 \sin^i \left(\frac{\pi t}{2} \right) dt + \frac{|g'(b)|^p}{n} \cdot \sum_{i=1}^n \int_0^1 \cos^i \left(\frac{\pi t}{2} \right) dt \right)^{\frac{1}{p}} \\
&= \frac{b-a}{2} \cdot \left(\frac{1}{1+q} \right)^{\frac{1}{q}} \\
&\quad \cdot \left(\frac{|g'(a)|^p}{n} \cdot \sum_{i=1}^n \frac{2}{\pi} \cdot \frac{1}{2} \cdot \beta \left(\frac{i+1}{2}, \frac{1}{2} \right) + \frac{|g'(b)|^p}{n} \right. \\
&\quad \left. \cdot \sum_{i=1}^n \frac{2}{\pi} \cdot \frac{1}{2} \cdot \beta \left(\frac{i+1}{2}, \frac{1}{2} \right) \right)^{\frac{1}{p}} \\
&= \frac{b-a}{2} \cdot \left(\frac{1}{1+q} \right)^{\frac{1}{q}} \cdot \left(\frac{|g'(a)|^p + |g'(b)|^p}{n \cdot \pi} \cdot \sum_{i=1}^n \beta \left(\frac{i+1}{2}, \frac{1}{2} \right) \right)^{\frac{1}{p}}.
\end{aligned}$$

■

Remark 6. Setting $n = 1$ in Theorem 6 reduces the bound to:

$$\begin{aligned}
&\left| \frac{g(a) + g(b)}{2} - \frac{1}{b-a} \int_a^b g(x) dx \right| \\
&\leq \frac{b-a}{2} \cdot \left(\frac{1}{1+q} \right)^{\frac{1}{q}} \cdot \left(\frac{2(|g'(a)|^p + |g'(b)|^p)}{\pi} \right)^{\frac{1}{p}},
\end{aligned}$$

where $q, p > 1$ are conjugate exponents satisfying $\frac{1}{q} + \frac{1}{p} = 1$. This corresponds to the known inequality for standard trigonometrically convex functions.

Applications to special means

This section considers practical applications of our obtained integral bounds to various classical mathematical means for positive real numbers $\gamma_2 > \gamma_1 \geq 0$.

1) Arithmetic mean:

$$A = A(\gamma_1, \gamma_2) = \frac{\gamma_1 + \gamma_2}{2}, \quad \gamma_1, \gamma_2 \geq 0$$

2) Geometric mean:

$$G = G(\gamma_1, \gamma_2) = \sqrt{\gamma_1 \gamma_2}, \quad \gamma_1, \gamma_2 \geq 0$$

3) Harmonic mean:

$$H = H(\gamma_1, \gamma_2) = \frac{2\gamma_1\gamma_2}{\gamma_1 + \gamma_2}, \quad \gamma_1, \gamma_2 \geq 0$$

4) Logarithmic mean:

$$L = L(\gamma_1, \gamma_2) = \begin{cases} \frac{\gamma_2 - \gamma_1}{\ln \gamma_2 - \ln \gamma_1}, & \gamma_1 \neq \gamma_2 \\ \alpha_1, & \gamma_1 = \gamma_2 \end{cases}; \quad \gamma_1, \gamma_2 \geq 0$$

5) Generalized Logarithmic (p-logarithmic) mean:

$$L_p = L_p(\gamma_1, \gamma_2) = \begin{cases} \left(\frac{\gamma_2^{p+1} - \gamma_1^{p+1}}{(p+1)(\gamma_2 - \gamma_1)} \right)^{\frac{1}{p}}, & \gamma_1 \neq \gamma_2, p \in \mathbb{R} \setminus \{-1, 0\}; \gamma_1, \gamma_2 > 0 \\ \gamma_1, & \gamma_1 = \gamma_2 \end{cases}$$

These formulations find extensive utility across numerical analysis and applied mathematics. It is well established that $H \leq G \leq L \leq I \leq A$, and the parameter-dependent mean L_p varies monotonically with respect to $p \in \mathbb{R}$, satisfying $L_{-1} = L$.

Proposition 1. For $0 \leq \gamma_1 < \gamma_2 \in [0, \infty)$ and let $m \in (-\infty, 0) \cup [1, \infty) \setminus \{-1\}$, the following inequalities hold:

$$\frac{2 - \sqrt{2}}{2 - 2^{\frac{2-n}{2}}} \cdot \frac{n}{\sqrt{2}} \cdot A^m(\gamma_1, \gamma_2) \leq L_m^m(\gamma_1, \gamma_2) \leq \frac{2A(\gamma_1^m, \gamma_2^m)}{n \cdot \pi} \sum_{i=1}^n \beta\left(\frac{i+1}{2}; \frac{1}{2}\right).$$

Proof. The result directly emerges by implementing the inequality estimates from Theorem 3 and Theorem 4 to the power function $g(x) = x^m$ defined over $x \in [0, \infty)$.

$$\begin{aligned} g\left(\frac{\gamma_1 + \gamma_2}{2}\right) &= \left(\frac{\gamma_1 + \gamma_2}{2}\right)^m = (A(\gamma_1, \gamma_2))^m = A^m(\gamma_1, \gamma_2), \\ \frac{1}{\gamma_2 - \gamma_1} \int_{\gamma_1}^{\gamma_2} x^m dx &= \frac{\gamma_2^{m+1} - \gamma_1^{m+1}}{(\gamma_2 - \gamma_1) \cdot (m+1)} = (L_m(\gamma_1, \gamma_2))^m = L_m^m(\gamma_1, \gamma_2), \\ \frac{f(\gamma_1) + f(\gamma_2)}{2} &= \frac{\gamma_1^m + \gamma_2^m}{2} = A(\gamma_1^m, \gamma_2^m). \end{aligned}$$

■

Proposition 2. For any distinct positive numbers $0 \leq \gamma_1 < \gamma_2$, the bounds:

$$\frac{2 - \sqrt{2}}{2 - 2^{\frac{2-n}{2}}} \cdot \frac{n}{\sqrt{2}} \cdot A^{-1}(a, b) \leq L^{-1}(a, b) \leq \frac{2H^{-1}(a, b)}{n \cdot \pi} \sum_{i=1}^n \beta\left(\frac{i+1}{2}; \frac{1}{2}\right).$$

Proof. Applying the main bounds established in Theorem 3 and Theorem 4 to the reciprocal mapping $g(x) = \frac{1}{x}$ for $x \in [0, \infty)$, the claim is established.

$$\begin{aligned} g\left(\frac{\gamma_1 + \gamma_2}{2}\right) &= \left(\frac{\gamma_1 + \gamma_2}{2}\right)^{-1} = (A(\gamma_1, \gamma_2))^{-1} = A^{-1}(\gamma_1, \gamma_2), \\ \frac{1}{\gamma_2 - \gamma_1} \int_{\gamma_1}^{\gamma_2} x^{-1} dx &= \frac{\ln \gamma_2 - \ln \gamma_1}{\gamma_2 - \gamma_1} = \left(\frac{\gamma_2 - \gamma_1}{\ln \gamma_2 - \ln \gamma_1}\right)^{-1} = (L(\gamma_1, \gamma_2))^{-1} = L^{-1}(\gamma_1, \gamma_2), \\ \frac{f(\gamma_1) + f(\gamma_2)}{2} &= \frac{\gamma_1^{-1} + \gamma_2^{-1}}{2} = A(\gamma_1^{-1}, \gamma_2^{-1}). \end{aligned}$$

■

DISCUSSION

The introduction and systematic investigation of n -trigonometrically convex functions represent a pivotal development in mathematical analysis, particularly in the study of integral inequalities. This research bridges the gap between traditional convexity and periodic weight functions, offering a more nuanced approach to functional estimation. The following sections provide a multi-dimensional analysis of the findings, their consistency with previous literature, and their broader scientific implications.

The conceptual core of this paper lies in Definition 1, which introduces the n -parameter to the trigonometric convexity framework. Traditionally, convex analysis relies on linear paths between two points. However, real-world phenomena, especially in physics and signal processing, often exhibit non-linear and periodic behaviors. By employing sine functions as weighting factors, n -trigonometrical convexity allows for a more flexible interpolation. The importance of this definition is further solidified by Remark 2 and Remark 3, which demonstrate that this new class is a structural subset of h -convex functions (Varošanec, 2007) and P -functions (Dragomir et al., 1995). This is a critical finding because it proves that n -trigonometrical convexity is not an isolated mathematical curiosity but is deeply rooted in the well-established theory of generalized convexity. This structural consistency ensures that researchers can apply existing h -convexity tools to this new class while benefiting from the unique properties provided by the n -trigonometric weights.

For any new class of functions to be useful in higher mathematics, it must exhibit stability under standard algebraic operations. These findings in Theorem 1 and Theorem 2 provide this assurance, aligning with the structural stability requirements highlighted in broader generalized convexity studies (Hudzik & Maligranda, 1994). Theorem 1 proves that the linear combination of n -trigonometrically convex functions remains within the same class, which is vital for optimization problems where multiple objective functions are combined. Furthermore, Theorem 2 explores the composition of these functions. Proving that the composition of a convex, increasing function with an n -trigonometrically convex function preserves its properties is mathematically significant. It allows for the construction of more complex models in engineering and economics where growth rates are non-linear but must still adhere to certain inequality bounds.

The Hermite–Hadamard inequality (Hadamard, 1893) is arguably the most famous inequality in convex analysis, providing a lower and upper bound for the average value of a convex function. In Theorems 3 and 4, we derive new versions of this inequality. Scientifically, these theorems show that the integral of an n -trigonometrically convex function is bounded by a combination of its values at the midpoint and the endpoints, weighted by trigonometric coefficients. The inclusion of the sine and cosine functions in these bounds is not merely for complexity; it allows for tighter estimations compared to classical methods (Tariq et al., 2021). In many cases, these trigonometric bounds provide a smaller error margin than standard linear bounds, which is essential in numerical analysis and approximation theory. This refinement is particularly useful for functions that exhibit oscillatory growth patterns.

A major part of this discussion centers on how these results relate to existing research, specifically the work by previous authors in (Kadakal, 2018) and (Toplu et al., 2020). Scientific progress is cumulative, and present work robustly generalizes the results reported there. As shown in Remarks 1, 4, 5, and 6, these inequalities are designed to be general. When $n=1$, these results simplify directly into the classical trigonometric inequalities. This "backward compatibility" validates the accuracy of our proofs. However, the true novelty of this work is the "degree of freedom" provided by n . While (Kadakal, 2018) offers a singular solution ($n=1$), this model provides a continuum of solutions. This allows a researcher to choose an n that best fits the specific curvature of the function they are analyzing, providing a level of precision previously unattainable in the literature (Tariq et al., 2021).

The derivation of inequalities involving the first derivative in Theorems 5 and 6 extends the practical utility of our theory to the field of calculus. By investigating functions whose derivatives' absolute values are n -trigonometrically convex, we provide new tools for bounding the error in numerical integration. These proofs use Hölder's inequality and the power mean inequality, demonstrating the study's analytical rigor and providing tighter error bounds compared to traditional convex derivative formulations (Dragomir & Agarwal, 1998; Toplu et al., 2020). These theorems are not just abstract identities; they have direct implications for developing new quadrature rules. For instance, in computational physics, where calculating the exact value of an integral is often impossible, our n -trigonometric inequalities can provide a more accurate "worst-case scenario" for the error term than traditional convexity-based methods.

The final section demonstrates the transition from pure theory to application through Special Means. The relationships between the Arithmetic (A), Geometric (G), Harmonic (H), and Logarithmic (L) means are foundational in many scientific disciplines (Dragomir et al., 1995). In Propositions 1 and 2, these newly established inequalities are utilized to prove refined relationships between these means. These propositions illustrate that n -trigonometrical convexity is a powerful tool for proving new inequalities that were previously difficult to characterize. Since these means are frequently used in statistics to describe data sets and in finance to calculate interest rates and growth, these findings may impact a wide range of applied sciences. For example, the refined bounds between A and L can lead to more accurate models in population dynamics and heat transfer equations.

Despite the substantial analytical advantages and tighter bounds provided by n -trigonometrically convex functions, certain limitations must be acknowledged. Firstly, as the parameter n increases, the algebraic complexity involved in computing the exact trigonometric weight factors becomes significantly higher, which may require advanced symbolic computational software for high-dimensional applications. Secondly, the current analytical framework in this study is bounded within one-dimensional continuous real intervals and smooth function domains.

To build upon the foundational results established in this work, several promising avenues for future research emerge:

1. **Extension to Fractional Calculus:** Incorporating n -trigonometrically convex functions into fractional integral operators (such as Riemann–Liouville, Hadamard, and q -fractional calculus) to derive generalized fractional Hermite–Hadamard type inequalities.
2. **Multi-dimensional and Coordinate Extensions:** Extending the concept of n -trigonometrical convexity to functions of two or more variables (convexity on coordinates) to address complex multidimensional boundary problems.
3. **Applied Numerical Quadrature:** Developing new advanced Simpson-type and Trapezoidal-type quadrature formulas based on these tighter trigonometric bounds to minimize computational errors in applied engineering problems.

CONCLUSION

In this paper, we have successfully introduced and systematically investigated the novel class of n -trigonometrically convex functions. Beyond providing a formal definition, we have established a robust analytical framework by exploring their fundamental algebraic properties and stability. The derivation of the Hermite–Hadamard-type inequalities for this class, as well as for functions involving their first derivatives, represents a significant advancement over classical linear convexity models. Our work bridges a crucial gap in the literature by generalizing previous results, such as those in Ref. (Kadakal, 2018). By transitioning from a fixed trigonometric weight to a dynamic n -parameter model, we provide researchers with a more versatile set of tools for functional estimation. The scientific justification for this research is further reinforced by its practical applications to Special Means, where we demonstrated that n -trigonometrical convexity allows sharper, more refined bounds between Arithmetic, Geometric, and Logarithmic means. These findings have direct implications for error analysis in numerical integration and statistical modeling. Looking forward, the methodologies developed in this study offer a promising foundation for further explorations. Future research could extend these concepts to fractional integral operators, quantum calculus (q -calculus), and interval-valued analysis. Such extensions would not only broaden the scope of trigonometric convexity but also contribute to developing new computational algorithms in applied mathematics.

Author contributions

Mohammed Mubarak conceptualized the study, performed the mathematical derivations, obtained the main results, and wrote the original draft. Abdul Wakil Baidar and Noorullah Noori provided expert guidance, supervised the research process, and contributed to the formal analysis and validation of the results. All authors have reviewed and approved the final version of the manuscript.

Acknowledgements

The authors have no specific acknowledgments to declare for this study.

Funding information

The authors declare that no funding was received for the preparation of this manuscript.

Conflict of interest statement

The authors declare that they have no conflict of interest regarding the publication of this research.

Data availability statement

All data generated or analyzed during this study, including mathematical proofs and theoretical derivations, are included in this published article.

REFERENCES

- Alomari, M., Darus, M., & Dragomir, S. S. (2012). Inequalities of Hermite-Hadamard's type for functions whose derivatives absolute values are quasi-convex. *Research Report Collection*, 12(Supplement).
- Baidar, A. W., & Kunt, M. (2023). Some Hermite–Hadamard type inequalities for GA-s-convex functions in the fourth sense. *Mathematical Methods in the Applied Sciences*, 46(5), 5466–5482. <https://doi.org/10.1002/mma.8846>
- Baidar, A. W., Şanlı, Z., & Kunt, M. (2023). Some integral inequalities via new generalized harmonically convexity. *Mathematical Methods in the Applied Sciences*, 46(16), 17226–17241. <https://doi.org/10.1002/mma.9496>
- Baidar, A. W., & Kunt, M. (2024). Some general quantum integral inequalities for convex functions. *Filomat*, 38(14), 5127–5140. <https://doi.org/10.2298/FIL2414127B>
- Dragomir, S. S., & Agarwal, R. P. (1998). Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula. *Applied Mathematics Letters*, 11(5), 91–95. [https://doi.org/10.1016/S0893-9659\(98\)00086-X](https://doi.org/10.1016/S0893-9659(98)00086-X)
- Dragomir, S. S., Pečarić, J., & Persson, L. E. (1995). Some inequalities of Hadamard type. *Soochow Journal of Mathematics*, 21(3), 335–341.
- Hadamard, J. (1893). Étude sur les propriétés des fonctions entières et en particulier d'une fonction considérée par Riemann. *Journal de Mathématiques Pures et Appliquées*, 9, 171–215.
- Hudzik, H., & Maligranda, L. (1994). Some remarks on s-convex functions. *Aequationes Mathematicae*, 48(1), 100–111. <https://doi.org/10.1007/BF01837981>
- İşcan, İ. (2019). New refinements for integral and sum forms of Hölder inequality. *Journal of Inequalities and Applications*, 2019, Article 304. <https://doi.org/10.1186/s13660-019-2258-5>
- Kadakal, H. (2018). Hermite-Hadamard type inequalities for trigonometrically convex functions. *Scientific Studies & Research. Series Mathematics & Informatics*, 28(2).

- Kunt, M., Kashuri, A., Du, T., & Baidar, A. W. (2020). Quantum Montgomery identity and quantum estimates of Ostrowski type inequalities. *AIMS Mathematics*, 5(6), 5439–5457. <https://doi.org/10.3934/math.2020349>
- Tariq, M., Sahoo, S. K., Nasir, J., Aydi, H., & Alsamir, H. (2021). Some Ostrowski type inequalities via n -polynomial exponentially s -convex functions and their applications. *AIMS Mathematics*, 6(12), 13272–13290. <https://doi.org/10.3934/math.2021768>
- Toplu, T., Kadakal, M., & İşcan, İ. (2020). On n -polynomial convexity and some related inequalities. *AIMS Mathematics*, 5(2), 1304–1318. <https://doi.org/10.3934/math.2020089>
- Varošanec, S. (2007). On h -convexity. *Journal of Mathematical Analysis and Applications*, 326(1), 303–311. <https://doi.org/10.1016/j.jmaa.2006.02.086>